ON POLYHEDRAL APPROXIMATION OF L-CONVEX AND M-CONVEX FUNCTIONS

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Abstract In discrete convex analysis, L-convexity and M-convexity are defined for functions in both discrete and continuous variables. Polyhedral L-/M-convex functions connect discrete and continuous versions. Specifically, polyhedral L-/M-convex functions with certain integrality can be identified with discrete versions. Here we show another role of polyhedral L-/M-convex functions: every closed L-/M-convex function in continuous variables can be approximated by polyhedral L-/M-convex functions, uniformly on every compact set. The proof relies on L-M conjugacy under the Legendre-Fenchel transformation.

Keywords: Discrete optimization, discrete convex analysis, L-convex function, M-convex function, polyhedral approximation

1. Introduction

In discrete convex analysis [4, 9, 10, 12], “convexity” concepts are defined for functions in both discrete and continuous variables. Specifically, three types of functions:

\[ f : \mathbb{Z}^n \to \mathbb{Z}, \quad f : \mathbb{Z}^n \to \mathbb{R}, \quad f : \mathbb{R}^n \to \mathbb{R} \]

are considered in discussing “convexity.” Furthermore, polyhedral and non-polyhedral (typically smooth) functions are distinguished for functions of type \( \mathbb{R}^n \to \mathbb{R} \). Set functions form a remarkable subclass of functions of type \( \mathbb{Z}^n \to \mathbb{Z} \) or \( \mathbb{Z}^n \to \mathbb{R} \).

L-convexity and M-convexity in discrete convex analysis are convexity concepts of combinatorial nature, defined for each of these classes of functions. \( L^b \)-convexity and \( M^b \)-convexity are variants of L-convexity and M-convexity, respectively. Submodular set functions are captured as \( L^b \)-convex functions of type \( \mathbb{Z}^n \to \mathbb{R} \), and matroids (basis families) are captured as \( M \)-convex functions of type \( \mathbb{Z}^n \to \mathbb{Z} \). L-convex functions of type \( \mathbb{Z}^n \to \mathbb{R} \) or \( \mathbb{R}^n \to \mathbb{R} \) find applications in operations research, queueing and inventory in particular (e.g., [1, 8, 20, 21]), through the equivalence between L-convexity and multimodularity [11]. M-convex functions play substantial roles in economics and game theory (e.g., [3, 5, 6, 17]) due to the equivalence between M-convexity and gross substitutes property.

Polyhedral L-/M-convex functions connect discrete and continuous versions in two directions: (i) convex extensions of L-/M-convex functions in discrete variables are (locally) polyhedral L-/M-convex functions in continuous variables, and (ii) discretization (or restriction to integer vectors) of polyhedral L-/M-convex functions with a certain integrality property results in L-/M-convex functions in discrete variables. Although polyhedral L-/M-convex functions are continuous functions of type \( \mathbb{R}^n \to \mathbb{R} \), they are endowed with combinatorial properties, sometimes called “discreteness in direction” [10].

In this paper we demonstrate another role of polyhedral L-/M-convex functions by establishing theorems stating that every closed L-/M-convex function in continuous variables

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can be approximated by polyhedral L-/M-convex functions, uniformly on every compact set. These theorems will serve to reinforce the connection between discrete and continuous versions of L-/M-convex functions.

As a motivation of the present work, a subtle technical aspect in polyhedral (or piecewise-linear) approximation of L-/M-convex functions is explained here. A standard technique of constructing a piecewise-linear convex approximation of a given function \( f : \mathbb{R}^n \to \mathbb{R} \) is to evaluate \( f(x) \) at finitely many sample points, say, \( x = x_1, \ldots, x_N \), and then take the convex lower envelope of the points \( (x_1, f(x_1)), \ldots, (x_N, f(x_N)) \) in \( \mathbb{R}^{n+1} \). A natural choice of the sample points for an L-/M-convex function \( f : \mathbb{R}^n \to \mathbb{R} \) is those points of \( (\frac{1}{k}\mathbb{Z})^n \) contained in a finite interval, where \( k \) is an integer. It can be shown that this standard technique basically works for L- or \( L^\natural \)-convex functions. However, it does not work for M- or \( M^\natural \)-convex functions. To be specific, a quadratic function \( f(x) = \frac{1}{2}x^\top Ax \) in \( x \in \mathbb{R}^3 \) with

\[
A = \begin{bmatrix}
3 & 2 & 1 \\
2 & 4 & 2 \\
1 & 2 & 3
\end{bmatrix}
\]

is an example of an \( M^\natural \)-convex function for which the standard procedure results in a piecewise-linear function that is not \( M^\natural \)-convex. We overcome this difficulty via conjugacy under the Legendre-Fenchel transformation. Given \( f \), we first consider its Legendre-Fenchel transform, say, \( g \). We apply the above-mentioned standard technique to \( g \) to obtain a piecewise-linear approximation, say, \( g_k \) to \( g \). We define \( f_k \) to be the Legendre-Fenchel transform of \( g_k \), and adopt \( f_k \) as a piecewise-linear approximation to \( f \). It can be shown that this method of construction works for M- or \( M^\natural \)-convex functions.

The rest of the paper is organized as follows. Section 2 offers preliminaries from discrete convex analysis, Section 3 presents the theorems (Theorems 3.1, 3.2 and 3.3) for L-convex functions, and Section 4 gives the corresponding results (Theorems 4.1 and 4.2) for M-convex functions.

2. Preliminaries

2.1. Convex functions

For a function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty, -\infty\} \), the effective domain and the epigraph are defined as

\[
\text{dom } f = \{x \in \mathbb{R}^n \mid -\infty < f(x) < +\infty\},
\]

\[
\text{epi } f = \{(x, y) \in \mathbb{R}^{n+1} \mid y \geq f(x)\}.
\]

The interior and the relative interior of the effective domain of \( f \) are denoted as \( \text{int } (\text{dom } f) \) and \( \text{ri } (\text{dom } f) \), respectively.

**Definition 2.1.** A function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is said to be convex if it satisfies the following inequality:

\[
\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda)y) \quad (0 \leq \lambda \leq 1).
\]

**Definition 2.2.** A convex function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is said to be proper if \( \text{dom } f \) is nonempty, and closed if \( \text{epi } f \) is a closed subset of \( \mathbb{R}^{n+1} \).

**Definition 2.3.** A function defined on \( \mathbb{R}^n \) is said to be polyhedral convex if its epigraph is a convex polyhedron in \( \mathbb{R}^{n+1} \). A polyhedral convex function is exactly such a function that can be represented as the maximum of a finite number of affine functions on a polyhedral effective domain.
Therefore, since
where (2.7) with
The inequality (2.7) with
special case of (2.7) with
Conversely, suppose that

Deﬁnition 2.6. A function
functions form a subclass of L
is nonnegative in (2.6), whereas it can be both negative and positive in (2.7), L-convex

Deﬁnition 2.5. A function
L-convex and L

A function

Proposition 2.1 ([15, Proposition 3.10]). A function g is L-convex if and only if it is a convex function that satisﬁes

\[ g(p) + g(q) \geq g((p - \alpha \mathbf{1}) \vee q) + g(p \wedge (q + \alpha \mathbf{1})) \quad (\alpha \in \mathbb{R}, \ p, q \in \mathbb{R}^n). \]  

Proof. * If g is an L-convex function, then

\[
g(p) + g(q) = g(p) + g(q + \alpha \mathbf{1}) - \alpha r \\
\geq g(p \vee (q + \alpha \mathbf{1})) + g(p \wedge (q + \alpha \mathbf{1})) - \alpha r \\
= g((p \vee (q + \alpha \mathbf{1})) - \alpha \mathbf{1} + g(p \wedge (q + \alpha \mathbf{1})) \\
= g((p - \alpha \mathbf{1}) \vee q) + g(p \wedge (q + \alpha \mathbf{1})).
\]

Conversely, suppose that g satisﬁes the inequality (2.7). Submodularity (2.4) follows as a special case of (2.7) with \( \alpha = 0 \). Linearity in direction \( \mathbf{1} \) in (2.5) can be derived as follows. The inequality (2.7) with \( p = q = s, \alpha = -\beta \leq 0 \) yields \( 2g(s) \geq g(s + \beta \mathbf{1}) + g(s - \beta \mathbf{1}) \), whereas (2.7) with \( p = s + \beta \mathbf{1}, q = s - \beta \mathbf{1}, \alpha = \beta \) yields \( g(s + \beta \mathbf{1}) + g(s - \beta \mathbf{1}) \geq 2g(s) \). Therefore,

\[ g(s + \beta \mathbf{1}) + g(s - \beta \mathbf{1}) = 2g(s) \quad (0 \leq \beta \in \mathbb{R}, \ s \in \mathbb{R}^n). \]

Since g is a convex function, this implies (2.5). \( \square \)

The inequality (2.7) is the same as (2.6) in form, but different in the range of \( \alpha \). Since \( \alpha \) is nonnegative in (2.6), whereas it can be both negative and positive in (2.7), L-convex functions form a subclass of \( L^2 \)-convex functions. Nevertheless, L-convex functions and \( L^2 \)-convex functions are essentially the same, in the sense that \( L^2 \)-convex functions in \( n \) variables

"The proof is given here as it is omitted in [15].

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Deﬁnition 2.4. A function is said to be locally polyhedral convex if it is a polyhedral convex function on any ﬁnite closed interval \([a, b]\) with \( a \leq b \).

See [7, 18] for more about convex functions.

2.2. L-convex functions

L-convex and \( L^2 \)-convex functions are deﬁned as follows.

Deﬁnition 2.5. A function \( g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is called L-convex if it is a convex function that satisﬁes the following two conditions:

• [Submodularity]:

\[ g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \quad (p, q \in \mathbb{R}^n), \]  

where \( p \vee q \) and \( p \wedge q \) are, respectively, the componentwise maximum and minimum of \( p \) and \( q \).

• [Linearity in direction \( \mathbf{1} \)]: There exists a real number \( r \) such that

\[ g(p + \alpha \mathbf{1}) = g(p) + \alpha r \quad (\alpha \in \mathbb{R}, \ p \in \mathbb{R}^n), \]  

where \( \mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n \).

Deﬁnition 2.6. A function \( g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is called \( L^2 \)-convex if it is a convex function that satisﬁes the following inequality:

\[ g(p) + g(q) \geq g((p - \alpha \mathbf{1}) \vee q) + g(p \wedge (q + \alpha \mathbf{1})) \quad (0 \leq \alpha \in \mathbb{R}, \ p, q \in \mathbb{R}^n). \]  

The property expressed by (2.6) is referred to as translation-submodularity.

The proof is given here as it is omitted in [15].
can be identified, up to the constant $r$ in (2.5), with L-convex functions in $n + 1$ variables [10].

$L^3$-convex functions in discrete variables are defined in terms of a discrete version of translation-submodularity.

**Definition 2.7.** A function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is called $L^3$-convex if it satisfies

$$g(p) + g(q) \geq g((p - \alpha 1) \lor q) + g(p \land (q + \alpha 1)) \quad (0 \leq \alpha \in \mathbb{Z}, \ p, q \in \mathbb{Z}^n).$$

(2.8)

### 2.3. M-convex functions

M-convex and $M^2$-convex functions are defined as follows. We denote by $\chi_i$ the $i$-th unit vector, i.e., $\chi_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ for $1 \leq i \leq n$, and the zero vector for $i = 0$, i.e., $\chi_0 = 0$. The positive and negative supports of a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ are denoted as

$$\text{supp}^+(x) = \{i \mid x_i > 0, \ 1 \leq i \leq n\}, \quad \text{supp}^-(x) = \{i \mid x_i < 0, \ 1 \leq i \leq n\}.$$ (2.9)

**Definition 2.8.** A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called $M$-convex if it is a convex function that satisfies the following exchange axiom:

*(M-EXC)* For any $x, y \in \mathbb{R}^n$ and any $i \in \text{supp}^+(x - y)$, there exists $j \in \text{supp}^-(x - y)$ and a positive real number $\alpha_0$ such that

$$f(x) + f(y) \geq f(x - \alpha (\chi_i - \chi_j)) + f(y + \alpha (\chi_i - \chi_j)) \quad (0 \leq \alpha \leq \alpha_0).$$

**Definition 2.9.** A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called $M^2$-convex if it is a convex function that satisfies the following exchange axiom:

*(M²-EXC)* For any $x, y \in \mathbb{R}^n$ and any $i \in \text{supp}^+(x - y)$, there exists $j \in \text{supp}^-(x - y) \cup \{0\}$ and a positive real number $\alpha_0$ such that

$$f(x) + f(y) \geq f(x - \alpha (\chi_i - \chi_j)) + f(y + \alpha (\chi_i - \chi_j)) \quad (0 \leq \alpha \leq \alpha_0).$$

Since $j = 0$ is allowed in $(M^2\text{-EXC})$ and not in $(M\text{-EXC})$, M-convex functions form a subclass of $M^2$-convex functions. Nevertheless, M-convex functions and $M^2$-convex functions are essentially the same, in the sense that $M^2$-convex functions in $n$ variables can be obtained as projections of M-convex functions in $n + 1$ variables [10].

### 2.4. Conjugacy

Conjugacy between L-convex functions and M-convex functions plays an important role in this paper. For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with $\text{dom} \ f \neq \emptyset$, the conjugate of $f$ is a function $f^\bullet : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^\bullet(p) = \sup \{\langle p, x \rangle - f(x) \mid x \in \mathbb{R}^n\} \quad (p \in \mathbb{R}^n),$$ (2.10)

where $\langle p, x \rangle$ denotes the standard inner product of two vectors $p$ and $x$. The function $f^\bullet$ is also called the Legendre–Fenchel transform of $f$, and the mapping $f \mapsto f^\bullet$ is referred to as the Legendre–Fenchel transformation.

**Theorem 2.2** ([14, Theorem 1.1]).

1. The classes of closed proper M-convex functions and closed proper L-convex functions are in one-to-one correspondence under the Legendre–Fenchel transformation (2.10). That is, if $f$ is a closed proper M-convex function and $g$ is a closed proper L-convex function, then $f^\bullet$ is a closed proper L-convex function, $g^\bullet$ is a closed proper M-convex function, $(f^\bullet)^\bullet = f$, and $(g^\bullet)^\bullet = g$.

2. The classes of closed proper $M^2$-convex functions and closed proper $L^2$-convex functions are in one-to-one correspondence under the Legendre–Fenchel transformation (2.10).
Polyhedral M-convex and L-convex functions are conjugate to each other.

**Theorem 2.3** ([13, Theorem 5.1], [10, Theorem 8.4]).

1. The classes of polyhedral M-convex functions and polyhedral L-convex functions are in one-to-one correspondence under the Legendre–Fenchel transformation (2.10).

2. The classes of polyhedral $M^\#$-convex functions and polyhedral $L^\#$-convex functions are in one-to-one correspondence under the Legendre–Fenchel transformation (2.10).

### 3. Approximation of L-convex Functions

#### 3.1. Theorems

**Theorem 3.1.**

1. If a sequence of $L^2$-convex functions $g_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \ (k = 1, 2, \ldots)$ converges to a function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at every point of $\mathbb{R}^n$, then $g$ is an $L^2$-convex function.

2. The same statement with “$L^2$-convex” replaced by “$L$-convex” also holds.

**Proof.** The proof is given in Section 3.2.1.

**Theorem 3.2.**

1. For any closed proper $L^2$-convex function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, there exists a nonincreasing sequence $\{g_k\}$ of polyhedral $L^2$-convex functions $g_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \ (k = 1, 2, \ldots)$ that converges to $g$ uniformly on every compact subset of $\text{ri} \ (\text{dom} \ g)$ (the relative interior of the effective domain of $g$). In particular, for each $p \in \text{ri} \ (\text{dom} \ g)$, we have $g(p) = \lim_{k \to \infty} g_k(p)$.

2. The same statement with “$L^2$-convex” replaced by “$L$-convex” also holds.

**Proof.** The proof is given in Section 3.2.2.

**Example 3.1.** The function $g$ defined by

$$g(p) = \begin{cases} \frac{1}{p+1} & (p > -1) \\ +\infty & \text{(otherwise)} \end{cases}$$

is a closed proper $L^2$-convex function ($n = 1$) with $\text{dom} \ g = (-1, +\infty)$. This function can be represented as the limit of a sequence of polyhedral $L^2$-convex functions that converges to $g$ uniformly on every compact subset of the interval $(-1, +\infty) = \text{ri} \ (\text{dom} \ g)$. This fact follows from Theorem 3.2.

**Example 3.2.** The function $g$ defined by

$$g(p) = \begin{cases} p \log p & (p > 0) \\ 0 & (p = 0) \\ +\infty & (p < 0) \end{cases}$$

is a closed proper $L^2$-convex function ($n = 1$) with $\text{dom} \ g = [0, +\infty)$. At the end point $p = 0$ of $\text{dom} \ g$, it has no subgradients. This function can be represented as the limit of a sequence of polyhedral $L^2$-convex functions that converges to $g$ uniformly on every compact subset of $\text{dom} \ g = [0, +\infty)$. To see this, consider the piecewise-linear function that interpolates $g$ at $\frac{1}{k}Z$ and let $g_k$ be its restriction to the interval $[0, k]$. Then each $g_k$ is a polyhedral $L^2$-convex function and the sequence $\{g_k\}$ converges to $g$ uniformly on every compact subset $S$ of $\text{dom} \ g = [0, +\infty)$. In particular, the sequence converges to $g$ uniformly on $S = [0, 1]$, say. But this fact does not follow from Theorem 3.2, since $S = [0, 1]$ is not contained in $\text{ri} \ (\text{dom} \ g)$.

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[1] The assumption means that for each $p \in \mathbb{R}^n$, the limit $\lim_{k \to \infty} g_k(p)$ exists in $\mathbb{R} \cup \{+\infty\}$ and $g(p) = \lim_{k \to \infty} g_k(p)$. In particular, the possibility of $g_k(p) \to -\infty$ is excluded.
In Theorem 3.2 above the convergence is established in \( \text{ri}(\text{dom } g) \), whereas in the next theorem (Theorem 3.3) we extend this to \( \text{dom } g \) under the assumption of compactness of \( \text{dom } g \).

**Theorem 3.3.**

1. Let \( g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a closed proper \( L^1 \)-convex function with compact effective domain \( \text{dom } g \). Then there exists a sequence\(^\dagger\) \( \{g_k\} \) of polyhedral \( L^1 \)-convex functions \( g_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) \((k = 1, 2, \ldots)\) that converges to \( g \) uniformly on \( \text{dom } g \), i.e.,

\[
\lim_{k \to \infty} \sup_{p \in \text{dom } g} |g_k(p) - g(p)| = 0. \tag{3.1}
\]

2. The same statement with "\( L^1 \)-convex" replaced by "\( L \)-convex" also holds.

**Proof.** The proof relies on Theorem 3.2. See Section 3.2.3.

**Example 3.3.** The function \( g \) defined by

\[
g(p) = \begin{cases} 
p^2 & (|p| < 1) \\
2 & (|p| = 1) \\
+\infty & (|p| > 1)
\end{cases}
\]

is a (non-closed) \( L^1 \)-convex function \((n = 1)\) with \( \text{dom } g = [-1, 1] \). This function cannot be equal to the uniform limit of a sequence of polyhedral \( L^1 \)-convex functions. This example shows the necessity of the closedness assumption on \( g \) in Theorem 3.3. We add that a pointwise convergent sequence of polyhedral \( L^1 \)-convex functions does exist. For example, let \( g_k \) be the piecewise-linear function that interpolates \( g \) at \( \frac{1}{k}\Z \); we have \( g_k(1) = g_k(-1) = 2 \) and \( g_k(i/k) = g_k(-i/k) = (i/k)^2 \) for \( i = 0, 1, \ldots, k - 1 \). Then \( \lim_{k \to \infty} g_k(p) = g(p) \) for each \( p \in [-1, 1] \).

**Remark 3.1.** Here are two remarks about Theorems 3.2 and 3.3. First, in Theorem 3.2 we have a nonincreasing sequence \( \{g_k\} \), but this may not be the case in Theorem 3.3. Second, it seems difficult to derive Theorem 3.2 from Theorem 3.3.

### 3.2. Proofs

We first recall a fundamental fact.

**Lemma 3.4.** The pointwise limit of convex functions is a convex function.

**Proof.** The proof is given for completeness. Assume that a sequence of convex functions \( g_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) \((k = 1, 2, \ldots)\) converges pointwise, and denote by \( g(p) \) the limit of \( g_k(p) \) for each \( p \), i.e., \( g(p) = \lim_{k \to \infty} g_k(p) \). It may be that \( g(p) = -\infty \) for some \( p \) or \( g(p) \equiv +\infty \). In the inequality

\[
\lambda g_k(p) + (1 - \lambda)g_k(q) \geq g_k(\lambda p + (1 - \lambda)q) \quad (0 \leq \lambda \leq 1)
\]

for the convexity of \( g_k \), we let \( k \to \infty \) with \( \lambda \) fixed, to obtain

\[
\lambda g(p) + (1 - \lambda)g(q) \geq g(\lambda p + (1 - \lambda)q) \quad (0 \leq \lambda \leq 1).
\]

Hence \( g \) is convex. \( \square \)

\(^\dagger\)Unlike in Theorem 3.2, this sequence \( g_k \) is not necessarily nonincreasing.
3.2.1. Proof of Theorem 3.1

Convexity of the limit function follows from Lemma 3.4 above. In addition, L^2-convexity and L-convexity of the limit function can be proved as follows.

(1) Each g_k, being L^2-convex, has translation-submodularity in (2.6), i.e.,

\[ g_k(p) + g_k(q) \geq g_k((p - \alpha 1) \lor q) + g_k(p \land (q + \alpha 1)) \quad (0 \leq \alpha \in \mathbb{R}, \ p, q \in \mathbb{R}^n). \]

By letting k \to \infty, we obtain translation-submodularity (2.6) for g.

(2) By a similar argument with the use of (2.7) in place of (2.6).

3.2.2. Proof of Theorem 3.2

We make use of the following general convergence theorem.

**Lemma 3.5** ([18, Th.10.8]). Let C be a relatively open convex set, and let f_1, f_2, \ldots be a sequence of finite convex functions on C. Suppose that the sequence converges pointwise on a dense subset of C, i.e., that there exists a subset C’ of C such that ClC’ \supseteq C and, for each x \in C’, the limit of f_i(x), f_2(x), \ldots exists and is finite. The limit then exists for every x \in C, and the function f, where

\[ f(x) = \lim_{k \to \infty} f_k(x), \]

is finite and convex on C. Moreover the sequence of f_1, f_2, \ldots converges to f uniformly on each closed bounded subset of C.

**Lemma 3.6.** Let g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} be an L^2-convex function, and p_0 \in \text{dom} g.

(1) [Discretization with 1/2^{k-1} mesh] For k = 1, 2, \ldots, define h_k : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} by

\[ h_k(q) = g(p_0 + \frac{q}{2^{k-1}}) \quad (q \in \mathbb{Z}^n). \]

Then h_k is an L^2-convex function in discrete variables.

(2) Let \hat{h}_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} be the convex extension (convex closure) of h_k, and define \hat{g}_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} by

\[ \hat{g}_k(p) = \hat{h}_k(2^{k-1}(p - p_0)), \quad \text{i.e.,} \quad \hat{g}_k(p_0 + \frac{q}{2^{k-1}}) = \hat{h}_k(q). \]

Then each \hat{g}_k is a locally polyhedral L^2-convex function that satisfies \hat{g}_k \geq g on \mathbb{R}^n. Moreover, the sequence (\hat{g}_k | k = 1, 2, \ldots) is monotone nonincreasing.

(3) Let g_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} be the restriction of \hat{g}_k onto D_k = \{p \in \mathbb{R}^n \mid |p(i) - p_0(i)| \leq k (i = 1, 2, \ldots, n)\}. Each g_k is a polyhedral L^2-convex function that satisfies g_k \geq g on \mathbb{R}^n. Moreover, the sequence (g_k | k = 1, 2, \ldots) is monotone nonincreasing.

(4) (g_k | k = 1, 2, \ldots) converges to g uniformly on every compact subset of ri (dom g).

**Proof.** (1) Obviously, h_k is endowed with the discrete translation-submodularity (2.8).

(2) It is known [10] that an L^2-convex function in discrete variables is convex-extensible, and its convex closure is a locally polyhedral L^2-convex function. Therefore, \hat{g}_k is a locally polyhedral L^2-convex function. The monotonicity is obvious.

(3) D_k is a bounded L^2-convex set, and an L^2-convex function remains to be L^2-convex when it is restricted to an L^2-convex set. Therefore, g_k is a polyhedral L^2-convex function. The monotonicity of \{g_k\} follows from the monotonicity of \{\hat{g}_k\} and the inclusion D_k \subseteq D_{k+1}.

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Lemma 3.7

(16, Theorem 1.2)

3.2.3. Proof of Theorem 3.3

On ri (dom $g$), there exists a bounded convex set $C$ that is open relative to the affine hull of dom $g$ and\footnote{We may assume that cl $C$ is a bounded $L^2$-convex set.}

$$S \subset C \subset cl C \subset ri (dom g).$$

By the construction of $g_k$, there exists an integer $k(C)$ such that dom $g_k \supseteq C$ for all $k \geq k(C)$. For $k \geq k(C)$, let $g_k^C$ denote the restriction of $g_k$ to $C$. Then $(g_k^C \mid k \geq k(C))$ is a sequence of finite convex functions on $C$ to which we apply Lemma 3.5 with

$$C' = \{ p \in C \mid 2^{k-1}p \in \mathbb{Z}^n \text{ for some } k \geq k(C), k \in \mathbb{Z} \}.$$

Note that $C'$ is a dense subset of $C$, i.e., cl $C' \supseteq C$.

For each $p \in C'$ there exists $k = k(p)$ such that $2^{k-1}p \in \mathbb{Z}^n$, where we may assume $k(p) \geq k(C)$. Since $g^C_k = g^C_{k(p)} = g(p)$ for all $k \geq k(p)$, the sequence $(g^C_k \mid k \geq k(C))$ converges pointwise on $C'$. The first half of Lemma 3.5 shows that for each $p \in C$, the limit $g^C(p) = \lim_{k \to \infty} g^C_k(p) = \lim_{k \to \infty} g_k(p)$ exists, and the function $g^C$ is a convex function, which is finite-valued on $C$. By the latter half of Lemma 3.5, the sequence $(g^C_k \mid k \geq k(C))$ converges to $g^C$ uniformly on each compact subset of $C$. Obviously, we have $g^C_k(p) = g(p)$ for $p \in C'$, and hence $g^C(p) = g(p)$ for $p \in C$, since a convex function is continuous in the relative interior of the effective domain. Therefore, $(g^C_k \mid k \geq k(C))$ converges to $g$ uniformly on every compact subset of $C$, and, in particular, on $S$. Thus we conclude that $(g_k \mid k = 1, 2, \ldots)$ converges to $g$ uniformly on $S$. \hfill $\Box$

Theorem 3.2 follows from Lemma 3.6 above.

Example 3.4. The function $g$ defined by

$$g(p) = \begin{cases} -\sqrt{2-p^2} & (|p| \leq \sqrt{2}) \\ +\infty & (|p| > \sqrt{2}) \end{cases}$$

is a closed proper $L^2$-convex function with dom $g = [-\sqrt{2}, \sqrt{2}]$. In the construction in Lemma 3.6 we may choose $p_0 = 0$ to obtain polyhedral $L^2$-convex functions $g_k$. Since $\sqrt{2} \notin$ dom $g_k$ and $g_k(\sqrt{2}) = +\infty$ for every $k$, the sequence $g_k(p)$ does not converge to $g(p)$ at $p = \sqrt{2} \in$ dom $g$. Thus $\{g_k\}$ does not converge to $g$ on dom $g$, although it certainly does on ri (dom $g$) = $(-\sqrt{2}, \sqrt{2})$.

3.2.3. Proof of Theorem 3.3

We first recall two fundamental facts that we use.

Lemma 3.7 ([16, Theorem 1.2]). A closed proper $L^2$-convex function is continuous on its effective domain.

Lemma 3.8 (Dini’s theorem, e.g., [2, Theorem 8.2.6], [19, Theorem 7.1.2]). If a monotone sequence of continuous functions on a compact set converges pointwise to a continuous function, then the convergence is uniform on the compact set.

In proving Theorem 3.3 we may assume that dom $g$ is full-dimensional, since otherwise, we may project it onto an appropriate coordinate plane while preserving $L^2$-convexity. For any positive number $a > 0$, define

$$g^a(p) = \min \{g(q) \mid \|p - q\|_\infty \leq a \}.$$  \hfill (3.2)

We consider a sequence $\{g^a\}$ by fixing a (strictly) decreasing sequence of $a$’s converging to zero; e.g., $a = 1/2, 1/2^2, 1/2^3, \ldots$. We shall first apply Theorem 3.2 to $g^a$ to obtain a
sequence of polyhedral $L^2$-convex functions $g_k^a$ ($k = 1, 2, \ldots$), and then extract a sequence $\tilde{g}_m$ ($m = 1, 2, \ldots$) from $\{g_k^a\}$ by choosing appropriate pairs $(a_m, k_m)$. Our construction is summarized as: $g \to g^a \to \tilde{g}_k^a \to \tilde{g}_m$.

The functions $g^a$ have the following properties.

1. Each $g^a$ is an $L^2$-convex function.

   (Proof) Let $\delta_S$ denote the indicator function of $S = \{p \in \mathbb{R}^n \mid \|p\|_\infty \leq a\}$. Then $\delta_S$ is a separable convex function, and $g^a$ is equal to the infimum convolution of $g$ and $\delta_S$. The infimum convolution of an $L^2$-convex function and a separable convex function is known to be $L^2$-convex.

2. $\text{dom } g^a = \text{dom } g + [-a1, a1]$ (Minkowski sum). In particular, $\text{int } (\text{dom } g^a) \supseteq \text{dom } g$.

3. The sequence $\{g^a\}$ is nondecreasing as $a \downarrow 0$. That is, $g^a(p) \leq g^b(p)$ if $a > b > 0$.

4. For each $p \in \text{dom } g$, the sequence $\{g^a(p)\}$ converges to $g(p)$ as $a \downarrow 0$, i.e.,

$$
\lim_{a \downarrow 0} g^a(p) = g(p) \quad (p \in \text{dom } g). \tag{3.3}
$$

(Proof) By Lemma 3.7, $g$ is continuous on $\text{dom } g$. Then (3.3) follows from the definition (3.2).

5. As $a \downarrow 0$, the sequence $\{g^a\}$ converges to $g$ uniformly on $\text{dom } g$, i.e.,

$$
\lim_{a \downarrow 0} \sup_{p \in \text{dom } g} |g^a(p) - g(p)| = 0. \tag{3.4}
$$

(Proof) The effective domain $\text{dom } g$ is a compact set by the assumption, and $g^a$ and $g$ are continuous on $\text{dom } g$ by Lemma 3.7. Moreover, as $a \downarrow 0$, the sequence $\{g^a\}$ is nondecreasing and converges pointwise to $g$, as shown above. Therefore, the convergence is uniform by Dini’s theorem (Lemma 3.8).

**Example 3.5.** For the function

$$
g(p) = \begin{cases} 
-\sqrt{2-p^2} & (|p| \leq \sqrt{2}), \\
+\infty & (|p| > \sqrt{2})
\end{cases}
$$

treated in Example 3.4, we have

$$
g^a(p) = \begin{cases} 
-\sqrt{2} & (|p| \leq a), \\
-\sqrt{2} - ((|p| - a)^2 & (a \leq |p| \leq \sqrt{2} + a), \\
+\infty & (|p| > \sqrt{2} + a),
\end{cases}
$$

and hence

$$
\sup_{p \in \text{dom } g} |g^a(p) - g(p)| = |g^a(\sqrt{2}) - g(\sqrt{2})| = \sqrt{2\sqrt{2}}a - a^2 \to 0 \quad (a \downarrow 0).
$$

For each $a > 0$ we apply Theorem 3.2 to $g^a$ to obtain a sequence of polyhedral $L^2$-convex functions $g_k^a$ ($k = 1, 2, \ldots$) that converges to $g^a$ on every compact set contained in $\text{ri } (\text{dom } g^a) = \text{int } (\text{dom } g^a)$. Since $\text{dom } g$ is a compact set contained in $\text{int } (\text{dom } g^a)$, we have

$$
\lim_{k \to \infty} \sup_{p \in \text{dom } g} |g_k^a(p) - g^a(p)| = 0. \tag{3.5}
$$

By (3.4), on the other hand, $\{g^a\}$ converges to $g$ uniformly on $\text{dom } g$ as $a \downarrow 0$, which implies that for any $\varepsilon > 0$, there exists $\tilde{a} = \tilde{a}(\varepsilon) > 0$ such that

$$
\sup_{p \in \text{dom } g} |g^a(p) - g(p)| < \varepsilon. \tag{3.6}
$$
By (3.5) for \( \hat{a} = \hat{a}(\varepsilon) \), there exists \( \hat{k} = \hat{k}(\varepsilon) \) such that
\[
\sup_{p \in \text{dom} g} |g_{\hat{k}}^\hat{a}(p) - g^{\hat{a}}(p)| < \varepsilon
\]
for all \( k \geq \hat{k} \). In particular, with \( k = \hat{k} \), we obtain
\[
\sup_{p \in \text{dom} g} |g_{\hat{k}}^\hat{a}(p) - g^{\hat{a}}(p)| < \varepsilon.
\]
A combination of (3.6) and (3.8) yields
\[
\sup_{p \in \text{dom} g} |g_{\hat{k}}^\hat{a}(p) - g(p)| \leq \sup_{p \in \text{dom} g} |g_{\hat{k}}^\hat{a}(p) - g^{\hat{a}}(p)| + \sup_{p \in \text{dom} g} |g^{\hat{a}}(p) - g(p)| < 2\varepsilon.
\]
By choosing \( \varepsilon = 1/m \) for \( m = 1, 2, \ldots \), we construct a sequence \( \{\tilde{g}_m\} \) as
\[
\tilde{g}_m = g_{\hat{k}(1/m)}^{\hat{a}(1/m)} \quad (m = 1, 2, \ldots).
\]

Then we have the following.
1. \( \text{dom} \tilde{g}_m = \text{dom} g_{\hat{k}(1/m)}^{\hat{a}(1/m)} \supseteq \text{dom} g \).
2. Each \( \tilde{g}_m \) is a polyhedral \( L^\circ \)-convex function.
3. \( \{\tilde{g}_m\} \) converges to \( g \) uniformly on \( \text{dom} g \).

(Proof) By (3.9) with \( \varepsilon = 1/m \) we have
\[
\sup_{p \in \text{dom} g} |\tilde{g}_m(p) - g(p)| < 2/m.
\]
Therefore,
\[
\lim_{m \to \infty} \sup_{p \in \text{dom} g} |\tilde{g}_m(p) - g(p)| = 0.
\]
The proof of Theorem 3.3 is completed.

4. Approximation of M-convex Functions

4.1. Theorems

**Theorem 4.1.**

(1) If a sequence of closed proper \( M^\circ \)-convex functions \( f_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) \((k = 1, 2, \ldots)\) converges to a function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) at every point of \( \mathbb{R}^n \), then \( f \) is an \( M^\circ \)-convex function (not necessarily closed)\(^*\).

(2) The same statement with “\( M^\circ \)-convex” replaced by “\( M \)-convex” also holds.

**Proof.** The proof is based on Theorem 3.2 and the conjugacy theorems (Theorems 2.2 and 2.3). See Section 4.2.1. \( \square \)

**Example 4.1.** Consider functions \( f_k(x) = \max(1 - kx, 0) \) with \( \text{dom} f_k = [0, 1] \). Each \( f_k \) is a closed proper \( M^\circ \)-convex function, and the limit
\[
\lim_{k \to \infty} f_k(x) = \begin{cases} 
1 & (x = 0), \\
0 & (0 < x \leq 1), \\
+\infty & (x \not\in [0, 1])
\end{cases}
\]
is an \( M^\circ \)-convex function, which is not closed.

\(^*\)The assumption means that for each \( x \in \mathbb{R}^n \), the limit \( \lim_{k \to \infty} f_k(x) \) exists in \( \mathbb{R} \cup \{+\infty\} \) and \( f(x) = \lim_{k \to \infty} f_k(x) \).

In particular, the possibility of \( f_k(x) \to -\infty \) is excluded.
Theorem 4.2.
(1) For any closed proper $M^i$-convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ there exists a nondecreasing sequence $\{f_k\}$ of polyhedral $M^i$-convex functions $f_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ ($k = 1, 2, \ldots$) that converges to $f$ uniformly on every compact subset of $\text{dom } f$. In particular, for each $x \in \text{dom } f$, we have $f(x) = \lim_{k \to \infty} f_k(x)$.
(2) The same statement with "$M^i$-convex" replaced by "$M$-convex" also holds.

Proof. The proof is given in Section 4.2.2.

Remark 4.1. Note that Theorem 4.2 asserts uniform convergence on every compact subset of $\text{dom } f$ (that may not be a subset of $\text{ri (dom } f\}$). Also note that no compactness assumption is imposed on $\text{dom } f$.

Remark 4.2. In applications, $M^i$-convex functions often appear as laminar convex functions, for which a polyhedral approximation can be constructed easily. By a laminar family we mean a nonempty family $T$ of subsets of $\{1, \ldots, n\}$ such that $A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$ for any $A, B \in T$. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called laminar convex if it can be represented as
$$f(x) = \sum_{A \in T} \varphi^A(x(A)) \quad (x \in \mathbb{R}^n)$$
for a laminar family $T$ and a family of univariate convex functions $\varphi^A : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ indexed by $A \in T$, where $x(A) = \sum_{i \in A} x_i$ for $x = (x_1, \ldots, x_n)$. A laminar convex function is $M^i$-convex.

To construct a polyhedral approximation of $f$, let $\hat{\varphi}^A_k$ be the piecewise-linear function that interpolates $\varphi^A$ at $\frac{1}{k} \mathbb{Z}$, and let $\varphi^A_k$ denote its restriction to the interval $[-k, k]$. Then the function $f_k$ defined by
$$f_k(x) = \sum_{A \in T} \varphi^A_k(x(A)) \quad (x \in \mathbb{R}^n)$$
is a polyhedral $M^i$-convex function, and the sequence $\{f_k\}$ converges (pointwise) to $f$. It is noted, however, that, unlike in Theorem 4.2, the sequence $\{f_k\}$ is nonincreasing and the convergence is not necessarily uniform on every compact subset of $\text{dom } f$.

4.2. Proofs

4.2.1. Proof of Theorem 4.1
It suffices to consider the case of $M$-convex functions. First recall from Lemma 3.4 that the limit of convex functions is a convex function.

To show (M-EXC) for $f$, take distinct $x, y \in \text{dom } f$ and $i \in \text{supp}^+(x - y)$. Since $f_k$ converges to $f$ pointwise, we have $x, y \in \text{dom } f_k$ for all sufficiently large $k$. Each $f_k$ is an $M$-convex function, and, by Lemma 4.3 below, there exists $j_k \in \text{supp}^-(x - y)$ such that
$$f_k(x) + f_k(y) \geq f_k(x - \alpha(\chi_i - \chi_j)) + f_k(y + \alpha(\chi_i - \chi_j)) \quad (0 \leq \alpha \leq \alpha_0),$$
where
$$\alpha_0 = \frac{x(i) - y(i)}{2|\text{supp}^-(x - y)|} > 0.$$Since $\text{supp}^-(x - y)$ is a finite set, there exists some $j \in \text{supp}^-(x - y)$ such that $j_k$ equals $j$ for infinitely many $k$. Fix such $j$ and take a subsequence $k_1 < k_2 < \cdots < k_l < \cdots$ with $j = j_{k_l}$ ($l = 1, 2 \ldots$). Then we have
$$f_{k_l}(x) + f_{k_l}(y) \geq f_{k_l}(x - \alpha(\chi_i - \chi_j)) + f_{k_l}(y + \alpha(\chi_i - \chi_j)) \quad (0 \leq \alpha \leq \alpha_0),$$
where \( \alpha_0 \) is independent of \( l \). Letting \( l \to \infty \) we obtain
\[
f(x) + f(y) \geq f(x - \alpha(x_i - x_j)) + f(y + \alpha(x_i - x_j)) \quad (0 \leq \alpha \leq \alpha_0),
\]
which shows (M-EXC) for \( f \).

**Lemma 4.3** ([14, Theorem 3.11]). Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a closed proper convex function. Then, \( f \) satisfies (M-EXC) if and only if it satisfies
\[(\text{M-EXC}) \quad \text{For any } x, y \in \text{dom } f \text{ and any } i \in \text{supp}^+(x - y), \text{ there exists } j \in \text{supp}^-(x - y) \text{ such that}
\]
\[
f(x) + f(y) \geq f(x - \alpha(x_i - x_j)) + f(y + \alpha(x_i - x_j)) \quad (0 \leq \alpha \leq \frac{x(i) - y(i)}{2|\text{supp}^-(x - y)|}).
\]

### 4.2.2. Proof of Theorem 4.2

Recall the notation (2.10) for the conjugate function:
\[
g^*(x) = \sup \{ \langle p, x \rangle - g(p) \mid p \in \mathbb{R}^n \} \quad (x \in \mathbb{R}^n).
\]
(4.1)

Our proof uses the following general facts about conjugate functions.

**Lemma 4.4** ([18, Corollary 12.2.2]). For any convex function \( g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \), we have
\[
g^*(x) = \sup \{ \langle p, x \rangle - g(p) \mid p \in \text{ri}(\text{dom } g) \} \quad (x \in \mathbb{R}^n).
\]
(4.2)

**Lemma 4.5.** Let \( g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) and \( g_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) \((k = 1, 2, \ldots)\) be convex functions with \( \text{dom } g \neq \emptyset \) and \( \text{dom } g_k \neq \emptyset \) \((k = 1, 2, \ldots)\). Assume that for each \( p \in \mathbb{R}^n \), the sequence \( \{g_k(p)\} \) is nonincreasing, bounded from below by \( g(p) \), i.e.,
\[
g_1(p) \geq g_2(p) \geq \cdots \geq g_k(p) \geq g_{k+1}(p) \geq \cdots \geq g(p) \quad (p \in \mathbb{R}^n),
\]
(4.3)
and that \( \{g_k\} \) converges to \( g \) pointwise on \( \text{ri}(\text{dom } g) \), i.e.,
\[
\lim_{k \to \infty} g_k(p) = \inf_k g_k(p) = g(p) \quad (p \in \text{ri}(\text{dom } g)).
\]
(4.4)

Also assume that \( g^* \) is continuous on \( \text{dom } g^* \). Then the following hold.

1. The sequence \( \{g_k^*\} \) is nondecreasing and converges to \( g^* \) pointwise on \( \text{dom } g^* \). That is, for each \( x \in \text{dom } g^* \), we have \( g_k^*(x) \leq g_{k+1}^*(x) \) and \( \lim_{k \to \infty} g_k^*(x) = g^*(x) \).

2. The sequence \( \{g_k^*\} \) converges to \( g^* \) uniformly on every compact subset of \( \text{dom } g^* \).

**Proof.** (1) It follows from the monotonicity (4.3) of \( g_k \) and
\[
g_k^*(x) = \sup \{ \langle p, x \rangle - g_k(p) \mid p \in \mathbb{R}^n \} \quad (x \in \mathbb{R}^n)
\]
(4.5)
that \( g_k^*(x) \leq g_{k+1}^*(x) \leq \cdots \leq g^*(x) \). Define
\[
h(x) = \sup_k g_k^*(x) = \lim_{k \to \infty} g_k^*(x) \quad (x \in \mathbb{R}^n),
\]
where \( h(x) \in \mathbb{R} \cup \{+\infty\} \).

[Proof of \( h(x) \leq g^*(x) \)] By (4.5) and (4.3) we have
\[
g_k^*(x) = \sup_{p \in \mathbb{R}^n} \{ \langle p, x \rangle - g_k(p) \} \leq \sup_{p \in \mathbb{R}^n} \{ \langle p, x \rangle - g(p) \} = g^*(x)
\]
(4.6)
for any $x \in \mathbb{R}^n$. Taking the supremum over $k$ and using the definition of $h(x)$, we obtain $h(x) \leq g^*(x)$. This implies, in particular, that $\{g_k^*(x)\}$ has a finite limit for $x \in \text{dom} g^*$.

[Proof of $h(x) \geq g^*(x)$] For $x \in \text{dom} g^*$ we have

$$
h(x) = \sup_k g_k^*(x) = \sup_k \left( \sup_{p \in \mathbb{R}^n} \{ \langle p, x \rangle - g_k(p) \} \right) = \sup_k \left( \sup_{p \in \mathbb{R}^n} \{ \langle p, x \rangle - g_k(p) \} \right).
$$

Each $g_k \in \text{dom} g^*$, and using the definition of $h(x)$, we have

$$
h(x) = \sup_{p \in \mathbb{R}^n} \{ \langle p, x \rangle - \inf_k g_k(p) \} \geq \sup_{p \in \text{ri}(\text{dom} g)} \{ \langle p, x \rangle - \inf_k g_k(p) \} = \sup_{p \in \text{ri}(\text{dom} g)} \{ \langle p, x \rangle - g(p) \} = g^*(x),
$$

where the last equality is due to (4.2) in Lemma 4.4.

(2) Let $\mathcal{S} \subseteq \text{dom} g^*$ be a compact set. The sequence $\{g_k^*\}$ is nondecreasing and converges to $g^*$ pointwise on $\mathcal{S}$, where $g^*$ is continuous by the assumption. Then, by Dini’s theorem (Lemma 3.8), $\{g_k^*\}$ converges to $g^*$ uniformly on $\mathcal{S}$.

The following two lemmas show properties specific to $M^3$-convex and $L^3$-convex functions.

**Lemma 4.6** ([16, Theorem 1.1]). A closed proper $M^3$-convex function is continuous on its effective domain.

**Lemma 4.7.** For a closed proper $L^3$-convex function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, define polyhedral $L^3$-convex functions $g_k : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, as in Lemma 3.6.

1. $(g_k^* \mid k = 1, 2, \ldots)$ is nondecreasing and converges to $g^*$ pointwise on $\text{dom} g^*$. That is, for each $x \in \text{dom} g^*$, we have $g_k^*(x) \leq g_{k+1}^*(x)$ and $\lim_{k \to \infty} g_k^*(x) = g^*(x)$.

2. $(g_k^* \mid k = 1, 2, \ldots)$ converges to $g^*$ uniformly on every compact subset of $\text{dom} g^*$.

3. Each $g_k^*$ is a polyhedral $M^3$-convex function.

**Proof.** (1) & (2) We have $g_1 \geq g_2 \geq \cdots \geq g$ on $\mathbb{R}^n$ by Lemma 3.6(3), and the sequence $\{g_k\}$ converges to $g$ pointwise on $\text{ri}(\text{dom} g)$ by Lemma 3.6(4). The conjugate function $g^*$ is a closed proper $M^3$-convex function by Theorem 2.2, and is continuous on $\text{dom} g^*$ by Lemma 4.6. Hence Lemma 4.5 applies.

(3) $g_k^*$ is a polyhedral $M^3$-convex function by the polyhedral version of M-L conjugacy theorem (Theorem 2.3).

We now begin the proof of Theorem 4.2. For a closed proper $M^3$-convex function $f$, its conjugate $g = f^*$ is a closed proper $L^3$-convex function and $f = g^*$ by Theorem 2.2. From this $g$ construct $g_k$ as in Lemma 3.6, and then define $f_k = g_k^*$. Then Lemma 4.7 shows that $f_k$ is a polyhedral $M^3$-convex function, and $f_k$ converges to $f$ uniformly on every compact subset of $\text{dom} f$. Our construction is summarized as follows:

<table>
<thead>
<tr>
<th>$\text{dom} \hat{g}_k \subseteq \text{dom} g$</th>
<th>$\text{dom} g_k : \text{bounded}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L : g \to \hat{g}_k \to g_k$</td>
<td>$M : f \to f_k \to \text{dom} f_k = \mathbb{R}^n$</td>
</tr>
</tbody>
</table>

**Remark 4.3.** Here is an alternative proof, due to Shinji Ito, of the pointwise convergence in Lemma 4.5(1). Since $g_k \geq g$ we have $\text{dom} g_k \subseteq \text{dom} g$. By the assumption (4.4), there exists some $k'$ such that $\text{aff}(\text{dom} g_k) = \text{aff}(\text{dom} g)$ and $\text{ri}(\text{dom} g_k) \subseteq \text{ri}(\text{dom} g)$ for all $k \geq k'$, where $\text{aff}(\cdot)$ means the affine hull. Then it follows from Lemma 4.4 that $g^*(x) = \sup\{\langle p, x \rangle - g(p) \mid p \in \text{ri}(\text{dom} g)\}$, $g_k^*(x) = \sup\{\langle p, x \rangle - g_k(p) \mid p \in \text{ri}(\text{dom} g)\}$. 

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Therefore,
\[
\lim_{k \to \infty} g_k^\star(x) = \sup_{k \geq k'} \left( \sup_{p \in \operatorname{ri}(\operatorname{dom} g)} \{ \langle p, x \rangle - g_k(p) \} \right)
\]
\[
= \sup_{p \in \operatorname{ri}(\operatorname{dom} g)} \left( \sup_{k \geq k'} \{ \langle p, x \rangle - g_k(p) \} \right)
\]
\[
= \sup_{p \in \operatorname{ri}(\operatorname{dom} g)} \{ \langle p, x \rangle - g(p) \} = g^\star(x).
\]

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