Heuristic and Exact Algorithms for the Interval Min–Max Regret Generalized Assignment Problem

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1. Introduction

Most optimization problems in real life do not have accurate estimates of the problem parameters at the optimization phase. Stochastic optimization and robust optimization are two approaches mostly used to solve optimization problems under uncertainty. The min–max regret approach is one of the typical approaches for robust optimization.

Regret is defined as the difference between the actual cost and optimal cost that would have been obtained if a different solution had been chosen. The purpose of the min–max regret approach is to minimize the worst-case regret.

In this paper, we consider the *generalized assignment problem* (GAP) with min–max regret criteria under interval cost. We propose two heuristic algorithms and two exact algorithms for the min–max regret generalized assignment problem with interval cost (MMR-GAP).

The first heuristic algorithm is based on an approach that solves GAP to optimality under a fixed scenario. We consider three scenarios: lowest cost, highest cost and median cost, and we show that solving the classical GAP under the median-cost scenario leads to a solution of MMR-GAP whose objective function value is within twice the optimal value. A more sophisticated approach is provided through a mixed integer programming (MIP) model by replacing the constraints regarding the optimal cost for the worst-case scenario with the dual of continuous relaxation.

The two exact algorithms are based on a basic result by Yaman et al. [1], which we call the *worst case lemma* in this paper.

The first exact algorithm, based on *Benders' decomposition*, solves a MIP model with incomplete scenarios and iteratively supplements those scenarios corresponding to violated constraints. The second approach is a branch-and-cut algorithm using Lagrangian relaxation in place of linear programming relaxation to provide tighter lower bounds. It also features an efficient variable fixing mechanism in which the time for computing lower bounds is reduced by using a two-direction dynamic programming approach.

A series of experiments on some benchmark instances is performed and we conclude with a comparison of the introduced algorithms.

2. Problem Description

The *generalized assignment problem* (GAP) is defined as follows. Given n jobs $J = \{1, \ldots, n\}$ and m agents $I = \{1, \ldots, m\}$, we undertake to determine a minimum cost assignment, subject to assigning each job to exactly one agent and satisfying a resource constraint for each agent. Assigning job j to agent i incurs a cost of c_{ij} and consumes an amount a_{ij} of a resource, whereas the total amount of the resource available at agent i (agent capacity) is b_i.

In many real-life situations, the cost c_{ij} is affected by many factors and can be unknown at the optimization stage. In this paper, we assume that cost c_{ij} varies within a given range $[c_{ij}^-, c_{ij}^+]$, which means that c_{ij} can take any value from this range. A vector of costs c_{ij}^s satisfying $c_{ij}^s \in [c_{ij}^-, c_{ij}^+]$ is called a *scenario* and is denoted by s. We define $z^s(x)$ to be the objective value of solution x under scenario s, while z_*^s denotes the optimal value under scenario s. The regret $r^s(x)$ associated with a solution x, under a scenario s, is the difference between these two values: $r^s(x) = z^s(x) - z_*^s$. The maximum regret $r_{\max}(x)$ of a solution x is the maximum $r^s(x)$ value over all scenarios. Then the *interval min–max regret generalized assignment problem* (MMR-GAP) is to find a feasible solution x such that the maximum regret is minimized.

We showed the following lemma about the complexity of MMR-GAP.

Lemma 2.1. *The decision version of MMR-GAP is Σ_2^p-complete.*

3. Heuristic Algorithms

3.1 Fixed-Scenario Algorithm

Because both GAP and MMR-GAP have the same set of feasible solutions, we can obtain a feasible solution to MMR-GAP by fixing the scenario and then solving the resulting GAP instance to optimality. We consider three basic scenarios: lowest cost $c_{ij}^s = c_{ij}^-$, highest cost $c_{ij}^s = c_{ij}^+$ and median cost $c_{ij}^s = (c_{ij}^- + c_{ij}^+)/2$.

For the median-cost scenario, we proved that a solution to MMR-GAP with approximation ratio at most 2 can be obtained by solving the GAP under the median-cost scenario. We denote by X_0 the set of all feasible solutions to MMR-GAP.

Lemma 3.1. *Let $\widetilde{s}$ be the median-cost scenario, i.e., $c_{ij}^{\widetilde{s}} = (c_{ij}^- + c_{ij}^+)/2$, $\forall i \in I$, $\forall j \in J$, and let $\widetilde{x}$ be an optimal solution to GAP under $\widetilde{s}$. Then, $r_{\max}(\widetilde{x}) \leq 2r_{\max}(x)$ holds for all $x \in X_0$.*

3.2 Dual Substitution Heuristic

The dual substitution heuristic can be generally explained as follows: Solve a MIP model in which the dual of the LP relaxation of GAP is used to represent an approximate value of the optimal cost for the worst-case scenario in regret calculation.

4. Exact Algorithm

Two exact algorithms proposed in this paper are both rooted from a MIP model of MMR-GAP. From the worst case lemma, introducing a new continuous variable ϕ, along with a constraint that forces ϕ to satisfy $\phi \leq z_*^s, \forall s \in S$, the MMR-GAP can be expressed by the following MIP model (MIP-MMR-GAP):

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^+ x_{ij} - \phi \tag{1}$$

$$\text{s.t. } \phi \leq \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij}^- + (c_{ij}^+ - c_{ij}^-)x_{ij})y_{ij}, \forall y \in X_0 \tag{2}$$

$$x \in X_0. \tag{3}$$

4.1 Benders' Decomposition

This paper presents a *Benders' decomposition* based algorithm, which solves, at each iteration, a *master problem* defined by a relaxation of MIP-MMR-GAP, in which we replace the set X_0 in (2) with X, where $X \subseteq X_0$, i.e., (2) is replaced with $\phi \leq \sum_{i=1}^m \sum_{j=1}^n (c_{ij}^- + (c_{ij}^+ - c_{ij}^-)x_{ij})y_{ij}, \forall y \in X$. Let (x^*, ϕ^*) be an optimal solution to the current master problem. The sub-problem is then used to find a violated constraint, which is unsatisfied by the current optimal solution to the master problem. If such a constraint is found, the algorithm adds the solution y corresponding to the violated constraint into X and solves the master problem again. Otherwise, the current optimal solution is an optimal solution of MMR-GAP.

4.2 Branch-and-Cut

Another approach we consider is a branch-and-cut algorithm. At each node of the search tree, we obtain a lower bound by solving the Lagrangian relaxation of MMR-GAP. For a fixed set of Lagrangian multipliers, this relaxation problem becomes m independent 0-1 knapsack problems, which we solve by dynamic programming (DP). This relaxation is further exploited in the variable fixing phase. By using two-direction DP tables, the time complexity for each agent i is reduced from $O(b_i mn^2)$ to $O(b_i mn)$.

5. Results and Conclusions

We performed computational experiments of all the algorithms in Section 3 and 4. Regarding the two heuristic algorithms, the dual substitution heuristic in most cases provides better upper bounds than the fixed-scenario heuristic. The comparison between two exact algorithms in Section 4 leads to the result that the Lagrangian-based branch-and-cut had better performance for instances of types A and E.

References

[1] H. Yaman, O. E. Karasan and M. C. Pinar, "The robust spanning tree problem with interval data," *Operations Research Letters*, **40**, 29–31, 2001.