# Centers of Generalized Complementarity Problems

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The central trajectory or the path of centers is known as an important conception which has been playing a substantial role in the development of interior-point methods for various problems such as linear programs, convex programs, monotone complementarity problems, semidefinite programs and semidefinite linear complementarity problems. Megiddo [6] showed that the central trajectory runs through the interior of the primal-dual feasible region of a LP (or the interior of the feasible region of a monotone LCP) and converges to a pair of primal-dual optimal solutions of the LP (or a solution of the LCP, resp.). Since then, there have been developed numerous interior-point algorithms based on the idea of "numerically tracing the central trajectory," and there have been done many theoretical studies on the existence and the continuity of the central trajectory for various problems [1, 5, 4]. The aim of the research is a unification and a further extension of some of those theoretical studies on the central trajectory to the monotone generalized complementarity problem (GCP).

Let $\mathcal{E}$ be a finite dimensional real vector space with an inner product $\langle x, y\rangle$ of $x,\ y \in \mathcal{E}$. A subset $\mathcal{F}$ of $\mathcal{E}\times\mathcal{E}$ is called *monotone* if $\langle x-x', y-y'\rangle \geq 0$ for every $(x,y),\ (x',y') \in \mathcal{F}$. A monotone subset $\mathcal{F}$ of $\mathcal{E}\times\mathcal{E}$ is called *maximal* if there is no monotone subset of $\mathcal{E}\times\mathcal{E}$ which properly contains $\mathcal{F}$.

Let $C$ be a full dimensional and closed pointed convex cone in $\mathcal{E}$ and $C^*$ its dual *i.e.*, $C^* := \{y \in \mathcal{E} : \langle x, y\rangle \geq 0 \text{ for all } x \in C\}$. Let $\mathcal{F}$ be a maximal monotone subset of $\mathcal{E}\times\mathcal{E}$. We deal with a monotone generalized complementarity problems in $\mathcal{E}$ [2, 3],

$$\text{GCP: Find an } (x,y) \in C \times C^* \text{ such that } (x,y) \in \mathcal{F} \text{ and } \langle x,y\rangle = 0. \qquad (1)$$

The monotone GCP (1) provides a unified mathematical model for various problems arising in different fields.

**Definition 0.1. (Definitions 2.1.1, 2.3.2 and 2.4.1 of [7]):** Let $\theta > 0$ and $F$ be a real valued function on Int $C$. $F$ is said to be *self-concordant* on Int $C$ if $F$ is $C^3$ convex function on Int $C$ and $|D^3F(x)[h,h,h]| \leq 2(D^2F(x)[h,h])^{\frac{3}{2}}$ for all $x \in \text{Int } C$ and all $h \in \mathcal{E}$. $F$ is called a *θ-logarithmically homogeneous self-concordant barrier* for $C$ if

$F$ is a self-concordant function on Int $C$,
$F(x^p) \to \infty$ for each sequence $\{x^p \in \text{Int } C\}$ that converges to a boundary of $C$,
$F(tx) = F(x) - \theta \log t$ for each $t > 0$ and $x \in \text{Int } C$.

A *modified Legendre transformation* $F^+ : \text{Int } C^* \to \mathcal{R}$ of a $\theta$-logarithmically homogeneous self-concordant barrier $F$ for $C$ is defined by $F^+(y) := \sup\{-\langle x,y\rangle - F(x) : x \in \text{Int } C\}$ for every $y \in \text{Int } C^*$.

In general we know that:

- Given an arbitrary full dimensional closed pointed convex cone $C$, there is a $\theta$-logarithmically homogeneous self-concordant barrier $F$ for $C$ with some $\theta \geq 1$ ([7, Theorem and Remark 2.5.1]).
- The modified Legendre transformation $F^+$ of a $\theta$-logarithmically homogeneous self-concordant barrier $F$ for $C$ is a $\theta$-logarithmically homogeneous self-concordant barrier for the dual cone $C^*$ of $C$ ([7, Theorem 2.4.1]).

Now we state our main theorem.

**Theorem 0.2.** *Let $F(\boldsymbol{x})$ be a $\theta$-logarithmically homogeneous self-concordant barrier function for $C$ and $F^+(\boldsymbol{y})$ its modified Legendre transformation. Assume that*

$$\mathcal{F} \cap (\text{Int } C \times \mathcal{E}) \neq \emptyset \ \ \textit{and}\ \mathcal{F} \cap (C \times \text{Int } C^*) \neq \emptyset. \tag{2}$$

*(i) For every $a > 0$, there is a unique minimizer $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}(a), \boldsymbol{y}(a))$ of the problem*

$$\left.\begin{array}{ll} \textit{minimize} & \langle \boldsymbol{x}, \boldsymbol{y}\rangle + a\,(F(\boldsymbol{x}) + F^+(\boldsymbol{y})) \\ \textit{subject to} & (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{F} \cap (\text{Int } C \times \text{Int } C^*)\,. \end{array}\right\} \tag{3}$$

*Moreover, $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}(a), \boldsymbol{y}(a))$ is a solution of the problem (3) if and only if*

$$(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{F} \cap (\text{Int } C \times \text{Int } C^*) \quad \textit{and} \quad -\boldsymbol{y} = aDF(\boldsymbol{x}). \tag{4}$$

*(ii) The solution set of the monotone GCP (1) is nonempty and compact.*

*(iii) $(\boldsymbol{x}(a), \boldsymbol{y}(a))$ is continuous in $a > 0$.*

*(iv) Let $\{a^p\}$ be an infinite sequence of positive numbers converging to 0. Then the sequence $\{(\boldsymbol{x}(a^p), \boldsymbol{y}(a^p))\}$ is bounded and every accumulation point of the sequence is a solution of the GCP (1).*

We call $(\boldsymbol{x}(a), \boldsymbol{y}(a))$ $(a > 0)$ a center and $\{(\boldsymbol{x}(a), \boldsymbol{y}(a)) \ : \ a > 0\}$ the central trajectory.

Theorem 0.2 generalizes some results on the existence and the continuity of the monotone central trajectory of the monotone complementarity problem (CP) [1, 5] and the monotone semidefinite complementarity (linear) problem (SD(L)CP) [4].

| | cone | dual cone | | log.hom.self-c. | mod.Legendre trans. | |
|---|---|---|---|---|---|---|
| **CP** | $\mathcal{R}^n_+$ | $\mathcal{R}^n_+$ | $n$ | $-\sum \log x_i$ | $-\sum \log y_i + c$ | $x_i y_i = a$ |
| **SDCP$^{(*)}$** | $\mathcal{S}^n_+$ | $\mathcal{S}^n_+$ | $n$ | $-\log\det \boldsymbol{X}$ | $-\log\det \boldsymbol{Y} + c$ | $\boldsymbol{XY} = a\boldsymbol{I}$ |
| **GCP** | $C$ | $C^*$ | $\theta$ | $F(\boldsymbol{x})$ | $F^+(\boldsymbol{y})$ | $\boldsymbol{y} = -aDF(\boldsymbol{x})$ |

(*): $\mathcal{S}^n_+$ is the class of $n \times n$-symmetric semidefinite matrices

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