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On the maximum balanced k-flow problem(Part 2)

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1. Introduction

The *maximum k-flow problem* in a network studied by D.K. Wagner and H. Wan [2] is a problem of finding a maximum k-flow in the network determining how much arc capacity to purchase for each arc and how much flow to send so as to maximize the net profit, where the capacity of any arc can be increased at a per-unit cost and every unit of flow sent from the source to the sink yields a payoff k. In our previous paper [1], we proposed the *maximum balanced k-flow problem* of finding a maximum k-flow in a network with an additional constraint described in terms of a balancing rate function, and showed a result stating that the proposed problem can be regarded as a generalization of the maximum balanced flow problem, provided that all the given data are rational. In this research, we prove that the above result is also valid for any given data.

2. Maximum balanced k-flows

We denote the sets of reals and of nonnegative reals by $\mathbb{R}$ and $\mathbb{R}_+$. Let $G=(V,A)$ be a directed graph with vertex set V and arc set A. We use $x(T)\equiv\sum_{t\in T}x(t)$ for a function x on A and $T\subseteq A$. Consider $N=(G=(V,A),u;s^+,s^-)$, where $u:A\rightarrow\mathbb{R}_+-\{0\}$ is capacity function on A, and s^+ and s^- ($s^+\neq s^-$) are the source and the sink of N. For a function $f:A\rightarrow\mathbb{R}$, define the *boundary* $\partial f:V\rightarrow\mathbb{R}$ of f by

$$\partial f(v)\equiv\sum_{a\in\delta^+v}f(a)-\sum_{a\in\delta^-v}f(a),\quad v\in V,$$

where δ^+v (resp. δ^-v) is the set of arcs in G which have v as their initial (resp. terminal) vertices. Given N, the *maximum flow problem* (MF) is as follows:

(MF): Maximize $\mathrm{val}_N(f)$ s.t.

$$\partial f(v)=0,\qquad v\in V-\{s^+,s^-\},\tag{1}$$

$$0\leq f(a)\leq u(a),\qquad a\in A,\tag{2}$$

where f satisfying (1) is a *flow* in N of *flow-value* $\mathrm{val}_N(f)\equiv\partial f(s^+)$. A flow f is *feasible* if f satisfies (2). A feasible flow f maximizing $\mathrm{val}_N(f)$ is a *maximum* flow in N. For $S\subseteq V$ such that $s^+\in S$ and $s^-\in V-S$, $K(S)\equiv\{(u,v)\in A:u\in S,v\in V-S\}$ is a *cut* of G. The *value* of a cut $K(S)$ in N is $v(K(S))\equiv\sum_{a\in K(S)}u(a)$. A *minimum* cut is a cut minimizing the value. Given a *balancing rate* function $\alpha:A\rightarrow\mathbb{R}_+-\{0\}$, the *maximum balanced flow problem* (MBF) in $N_1=(G=(V,A),u,\alpha;s^+,s^-)$ is:

(MBF): Maximize $\mathrm{val}_{N_1}(f)$ s.t. (1), (2) and

$$f(a)\leq\alpha(a)\mathrm{val}_{N_1}(f),\quad a\in A.\tag{3}$$

A flow f satisfying (3) is *balanced* in N_1. A feasible balanced flow f maximizing $\mathrm{val}_{N_1}(f)$ is a *maximum* balanced flow in N_1. Given *cost* function $c:A\rightarrow\mathbb{R}_+$ and a number $k>0$, the *maximum k-flow problem* $(\mathrm{MF})^c_k$ in $N_2=(G=(V,A),u,c,k;s^+,s^-)$ studied by Wanger and Wan ([2]) is as follows:

$(\mathrm{MF})^c_k$: Maximize $\mathrm{val}_{N_2}(k,c,f)\equiv k\mathrm{val}_{N_2}(f)-\sum_{a\in A}c(a)\max\{f(a)-u(a),0\}$ s.t. (1).

A flow f is a *k-flow* in N_2. A k-flow f maximizing $\mathrm{val}_{N_2}(k,c,f)$ is a *maximum* k-flow in N_2. Given $N_3=(G=(V,A),u,\alpha,c,k;s^+,s^-)$, consider the *maximum balanced k-flow problem* $(\mathrm{MBF})^c_k$ defined as follows:

$(\mathrm{MBF})^c_k$: Maximize $\mathrm{val}_{N_3}(k,c,f)$ s.t. (1) and

$$f(a)\leq\alpha(a)\mathrm{val}_{N_3}(k,c,f),\quad a\in A.\tag{4}$$

A flow f satisfying (4) is a *balanced k-flow* in N_3. A balanced k-flow f maximizing $\mathrm{val}_{N_3}(k,c,f)$ is a *maximum* balanced k-flow in N_3. We introduce a parametric problem $(\mathrm{MF})_y$ in $N_y = (G=(V,A),\alpha,u;s^+,s^-,y)$ with parameter y:

$(\mathrm{MF})_y$: Maximize $\mathrm{val}_{N_y}(f)$ s.t. (1) and
$$0 \le f(a) \le \min\{u(a),\alpha(a)y\}, \quad a \in A. \quad (5)$$

Let $y^* = \max\{y' : y' = \mathrm{val}_{N_{y'}}(f_{y'})\}$, where $f_{y'}$ is a maximum flow in $N_{y'}$. For a minimum cut $K(S_{y^*})$ in N_{y^*}, we have

$$y^* = u(K^1(S_{y^*}))/(1-\alpha(K^2(S_{y^*}))), \quad (6)$$

where $K^1(S_{y^*}) = \{a \in K(S_{y^*}) : u(a) \le \alpha(a)y^*\}$ and $K^2(S_{y^*}) = K(S_{y^*}) - K^1(S_{y^*})$.

3. Analysis

We show that $(\mathrm{MBF})_1^c$ can be regarded as a generalization of (MBF). Let $u_\epsilon(a) = u(a) + \epsilon(a)$ for any given nonnegative numbers $\epsilon(a)$ $(a \in A)$. Then the optimal value y_ϵ^* in $N_y^\epsilon = (G,u_\epsilon,\alpha;s^+,s^-,y)$ is given by

$$y_\epsilon^* = u_\epsilon(K^1(S_{y_\epsilon^*}))/(1-\alpha(K^2(S_{y_\epsilon^*}))) \ , \quad (7)$$

where $K(S_{y_\epsilon^*})$ is a minimum cut in $N_{y_\epsilon^*}^\epsilon$ and $K^2(S_{y_\epsilon^*}) = K(S_{y_\epsilon^*}) - K^1(S_{y_\epsilon^*})$. Note that y_ϵ^* (resp. y^*) is the flow-value of a maximum balanced flow in $N_1^\epsilon = (G=(V,A),u_\epsilon,\alpha;s^+,s^-)$ (resp. N_1). Let $\Gamma = \min\{1-\alpha(B) : 1-\alpha(B) > 0, B \subseteq A\}$, and the following lemma holds.

Lemma 1: *Let $c(a) > C_1 \equiv m/\Gamma$ for any $a \in A$ and f_ϵ^* be a maximum balanced flow in N_1^ϵ. Then we have $y^* > \mathrm{val}_{N_3}(1,c,f_\epsilon^*)$ if*

$$y^* \ge u(K^1(S_{y_\epsilon^*}))/(1-\alpha(K^2(S_{y_\epsilon^*}))),$$

where $m = |A|$. □

In the following, we consider the case when

$$y^* < u(K^1(S_{y_\epsilon^*}))/(1-\alpha(K^2(S_{y_\epsilon^*}))). \quad (8)$$

It is easy to see the following lemma.

Lemma 2: *We have $y_\epsilon^* - y^* \le C_2\Delta^{**}$, where $U_{\max} = \max_{a\in A} u(a)$, $U_{\min} = \min_{a\in A} u(a)$, $\Delta^{**} = \max_{a\in A}\epsilon(a)$ and $C_2 = (mU_{\max})/U_{\min}$.* □

Lemma 3: *If we have (8) and $\delta^{**} \equiv \max_{a\in A}(f_\epsilon^*(a) - u(a)) < \Delta^{**}$, then there exists $(\eta(a) : a \in A)$ satisfying the three following conditions:*

(i) $0 \le \eta(a) \le \epsilon(a) \quad (a \in A)$,

(ii) f_ϵ^* *is a maximum balanced flow in* N_1^η,

(iii) $\delta^{**} = \max_{a\in A}\eta(a)$. □

From lemmas 2 and 3, we have

Lemma 4: *Let $c(a) > C_2$ for any $a \in A$. If the inequality (8) holds, then we have*

$y^* > \mathrm{val}_{N_3}(1,c,f_\epsilon^*)$. □

Consider N_3, where $c(a) > \max\{C_1, C_2\}$ for each $a \in A$. Let f_1 be any balanced 1-flow in N_3. From $c(a) > 0$ $(a \in A)$ and (4) we have $f_1(a) \le \alpha(a)\mathrm{val}_{N_3}(f_1)$, which implies that f_1 is a balanced flow in N_1. Define ϵ by

$$\epsilon(a) = \begin{cases} 0 & (f_1(a) \le u(a)\ ), \\ f_1(a) - u(a) & (\text{otherwise}). \end{cases}$$

Then f_1 is a feasible balanced flow in N_1^ϵ. From this, we have $\mathrm{val}_{N_3}(f_1) \le \mathrm{val}_{N_1^\epsilon}(f_\epsilon^*)$. From $\max\{f_1(a) - u(a), 0\} \ge \max\{f_\epsilon^*(a) - u(a), 0\}$ for any $a \in A$, we have $\mathrm{val}_{N_3}(1,c,f_1) \le \mathrm{val}_{N_3}(1,c,f_\epsilon^*)$. From lemmas 1 and 4, we have $\mathrm{val}_{N_3}(1,c,f_1) < y^*$. Hence we have the following theorem.

Theorem 5: *The maximum balanced k-flow problem is a generalization of the maximum balanced flow problem.* □

4. References

[1] A. Nakayama: On the maximum balanced k-flow problem, Faculty of Administrations and Social Sciences, Fukushima University,June,1995.

[2] D.K. Wagner and H. Wan: A polynomial-time simplex method for the maximum k-flow problem, *Mathematical Programming*,Vol.60, No.1 (1993), 115-123.