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A Polynomial Algorithm For Enumerating Vertices Of A Base Polyhedron

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1. Introduction and Definitions

It is well known that it is difficult to enumerate all vertices of a polytope in polynomial time. In this paper we will give a polynomial algorithm which enumerates vertices of a base polyhedron.

It should be noted that the result of our paper is a generalization of spanning tree enumeration problem and matroid base enumeration problem.

Given a finite set $N = \{1, 2, \cdots, n\}$, define a submodular function f: $2^N \to \mathbb{R}$ with

$$\forall X, Y \subseteq N : f(X)+f(Y) \geq f(X \cup Y)+f(X \cap Y).$$

The problem is to enumerate all vertices of the following base polyhedron

$$\mathrm{B}(f) = \{ x \mid x \in \mathbb{R}^N, \forall X \subset N : x(X) \leq f(X), \quad x(N) = f(N)\},$$

where, $x(X) = \sum_{e \in X} x(e)$.

Theorem 1.1: *A vector v is a vertex of a base polyhedron of* B(f) *if and only if for a maximal chain*

$$C : \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E$$

we have

$$v(j) = f(S_i) - f(S_{i-1}) \ (i = 1, 2, \cdots, n),$$

where $\{j\} = S_i - S_{i-1}$.

Here we introduce briefly certain lattice and poset related to a vertex of B(f). Let v be a vertex of B(f), define

$$\mathcal{D}(v) = \{ X \mid X \subseteq N, \ v(X) = f(X) \}.$$

Now we describe how to construct poset $P(\mathcal{D}(x))$. For a vertex v of a base polyhedron and an element i, define

$$\mathrm{dep}(v, i) = \bigcap \{X \mid i \in X \subseteq N, \ v(X) = f(X)\},$$

and define $j \preceq_v i$ if and only if $j \in$dep(v, i). We can draw the Hasse diagram $P(\mathcal{D}(v))$ as following. The vertex set of G_v is N, we draw an arc from i to j if and only if i covers j, i.e., $j \prec_v i$, and there is no $k (\neq i, j)$ such that $j \prec_v k \prec_v i$.

Theorem 1.2: *A vertex u is adjacent to v if and only if u is computed by a chain which is a chain of v but exchanging i and j of a pair (i, j), here (i, j) is an edge of G_v.*

2. Preliminary Results

We use G_v $(v \in V)$ to denote the Hasse diagram of a vertex v of a base polyhedron in the following. In the reverse search algorithm, let $v^* = (v_1^*, v_2^*, \cdots, v_n^*)$ be the optimal vertex of a base polyhedron, which is generated by $v_i^* = f(\{n, n-1, \cdots, i\}) - f(\{n, n-1, \cdots, i+1\})$ $(i = n-1, \cdots, 1)$, $v_n = f(n)$. Define the local search function f_B: $V \backslash v^* \to V$

$$f_B(v) := v + \tilde{c}(v, i^*, j^*)(\chi_{j^*} - \chi_{i^*})$$

where $i^* = \min\{i \mid j$ covers i in G_v and $i < j\}$, $j^* = \max\{j \mid j$ covers $i^*\}$ and $\tilde{c}(v, i^*, j^*)$ is a function of v, i^*, j^*.

Lemma 2.1: *Assume u is a vertex of* B(f) *and j covers i in G_u, and v is the vertex of* P(f) *which is obtained by $v = u + \tilde{c}(u, j, i)(\chi_j - \chi_i)$, let S_k' and S_k be the sets which are satisfied with*

$$S_k' = \bigcap \{T \mid u(T + i + j + k) = f(T + i + j + k)\}$$
$$S_k = \bigcap \{T \mid v(T + j + i + k) = f(T + j + i + k)\},$$

then we have $S_k' = S_k$ and also $S_k =$ dep$(v, i) \cup$ dep$(v, j) \cup$ dep$(v, k) - i - j - k$.

*Also let T=*dep$(v, i) \cup$dep(v, j), *then* $\tilde{c}(u, j, i) = -\tilde{c}(v, i, j) = f(T + j) + f(T + i) - f(T) - f(T + i + j) > 0$.

For brevity, we omit index of S_k, i.e., $S_k = S$.

Lemma 2.2: *Let v be a vertex of a base polyhedron and G_v its Hasse diagram. Exchange i and j of a pair (i,j) with i covering j in G_v to create a new adjacent vertex, say u. Then j with $i < j$ is the maximum index among the elements which cover i if and only if for pair (i,j) with $i < j$ and i covering j in G_v, there is no k in G_v which satisfies:*

(1) *k covers j with $k > j$ and*

$$f(S+i+j)+f(S+i+k) = f(S+i)+f(S+i+j+k),$$

and also no k in G_v which satisfies:

(2) *k covers i with $k > j$ and*

$$\begin{aligned} & f(S+i+j)+f(S+i+k) \\ &= f(S+i)+f(S+i+j+k), \\ & \qquad v(S) = f(S), \end{aligned}$$

where $S = \bigcap\{T \mid v(T+j+i+k) = f(T+j+i+k)\}$.

Lemma 2.3: *Let v be a vertex of a base polyhedron and G_v its Hasse diagram. Exchange i and j of a pair (i,j) with $j > i$ and i covering j in G_v to create a new adjacent vertex u. Then there is no pair (m,l) with $m > l$, $l < i$ and l covering m in G_u if and only if there is no pair (k,l) in G_v, that k covers l with $k > l$, here k may be m.*

3. An Enumerating Algorithm

For a pair (i,j) with $j > i$, we say it is false if in some Hasse diagram there is no such pair (i,j) with i covering j or there is a k which satisfies the condition of Lemma 2.2 . Otherwise, we say the pair (i,j) is true. Now we give the enumerating algorithm.

Input: A finite set $N = \{1,2,\cdots,n\}$, a submodular function f: $2^N \to \mathbf{R}$.
Output: All vertices of the base polyhedron.
Step 0: Assume the optimum vertex of base $v^* = (v_1^*, v_2^*, \cdots, v_n^*)$, where $v_i^* = f(\{n, n-1,\cdots,i\}) - f(\{n, n-1, \cdots, i+1\})$ $(i = n-1,\cdots,1)$, $v_n = f(n)$. Compute the Hasse diagram of the vertex v^* and $\mathrm{dep}(v^*,i)$ for each $i = (1,2,\cdots,n)$. Compute $l := \min\{i \mid j$ covers i and $i<j\}$. Get output v^*. Put $v := v^*$; $i := l$; $j := i+1$. Go to Step 1.
Step 1: (reverse search) If the pair (i,j) is true, let $T = \mathrm{dep}(v,i) \cup \mathrm{dep}(v,j) - i - j$, compute $\tilde{c}(v,i,j) = f(T+i)+f(T+j)-f(T+i+j)-f(T)$, get output $u := v + \tilde{c}(v,i,j)(\chi_j - \chi_i)$, put $v := u$, compute the Hasse diagram of the vertex v and $\mathrm{dep}(v,i)$ for each $i = (1,2,\cdots,n)$, and also compute $l := \min\{i \mid j$ covers i and $i < j\}$, put $i := l$; $j := i+1$
Otherwise, if $j < n$, put $j := j+1$.
Otherwise, if $i > 1$, put $i := i-1$ and $j := i+1$. Go to the beginning of Step 1.
Otherwise, if $v = v^*$, stop. Otherwise, go to Step 2.
Step 2: (forward traverse) Let $i^* = \min\{i \mid j$ covers i and $i < j\}$, and $j^* = \max\{j \mid i^*$ covers $j\}$.
If $j^* < n$, Put $i := i^*$ and $j := j^*+1$.
Otherwise, if $i^* > 1$, put $i := i^*-1$ and $j := i+1$. Let $T = \mathrm{dep}(v,i) \cup \mathrm{dep}(v,j) - i - j$, compute $\tilde{c}(v,j,i) = f(T+i)+f(T+j)-f(T+i+j)-f(T)$, put $u := v+\tilde{c}(v,j,i)(\chi_i-\chi_j)$, put $v := u$, compute the Hasse diagram of the vertex v and $\mathrm{dep}(v,i)$ for each $i = (1,2,\cdots,n)$. Go to the beginning of Step 1.
Otherwise, if $v = v^*$, stop. Otherwise, go to the beginning of Step 2.
(End)

Theorem 3.1: *There is an implementation of reverse search for enumerating vertices of a base polyhedron with time complexity $O(n^3|V|)$ and space complexity $O(n^2)$.*

References

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