Double Horn Functions

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1 Introduction

Boolean functions play important roles in many fields such as artificial intelligence, database theory and operations research. In particular, classes of *positive* (also called *monotone*) and *Horn* functions are ones of the most important classes of Boolean functions, where a function f is positive if it can be represented by a DNF (disjunctive normal form) in which each term contains no negative literal, and *Horn* if it can be represented by a DNF in which each term contains at most one negative literal. For example, positive functions are used to derive the reliability of systems, to construct coteries for mutual exclusion in distributed systems and to compute the power indices (e.g., *Shapley-Shubik index*) of the players of simple games. Also, Horn functions are widely used in practice as the production rules in expert systems, as basis of logic programming and as representation tools of functional dependencies in relational database systems. From these practical points, the various problems associated with these classes have been extensively studied.

It is well-known [1] that a Boolean function f is Horn if and only if $F(f)$ (set of false vectors of f) is closed under intersection (i.e., $v, w \in F(f)$ implies $v \bigwedge w \in F(f)$, where $\bigwedge$ denotes the componentwise AND operation). From a conceptual point, we would like to have a more balanced role between $T(f)$ and $F(f)$, where $T(f)$ is set of true vectors of f. A natural and suggestive possibility is to require that $T(f)$ and $F(f)$ behave dually, since 0 and 1 are dual values. This is the concept of bidual Horn functions: A function f is *bidual Horn* if $F(f)$ is closed under intersection and, dually, $T(f)$ is closed under union (i.e., under disjunction of vectors). Recalling that the dual f^d of f is defined by $f^d(x) = \overline{f}(\overline{x})$, this is equivalent to saying that f and f^d are both Horn. Therefore, the definition requires that both the sets $T(f)$ and $F(f)$ can be characterized by Horn rules, which is important in expert systems, for example, because a statement can always be either proved or disproved by applying Horn rules.

We can easily verify that positive functions are bidual Horn. Thus bidual Horn functions lie between two classes of positive and Horn functions. Besides these approaches, other possibilities also exist to balance to role of T and F. For example, the classes of submodular functions and double Horn functions have been investigated [3, 2], where a function f is submodular if f and the contra-dual f^* of f are Horn, and double Horn if f and $\overline{f}$ are Horn.

In this paper, we introduce bidual Horn functions, and characterize bidual Horn functions in terms of properties of their Horn implicants. Furthermore, we consider recognition problem and problem of computing term-shortest and literal-shortest DNF representations, where DNF φ is called *term-shortest* (resp., *literal-shortest*) if there is no DNF containing fewer terms (resp., a smaller total number of literals), which represents the same function. These DNFs reduce the memory requirements and increases the computational efficiency of Horn based expert system.

2 Preliminaries

For vectors $v, w \in \{0,1\}^n$, let $v \wedge w$ (resp., $v \vee w$) denote the *intersection* (resp., *union*) (i.e., the componentwise conjunction (resp., disjunction)) of vectors v and w, e.g., if $v = (1100)$ and $w = (1010)$, then $v \wedge w = (1000)$ and $v \vee w = (1110)$. Recall that a *Boolean function*, or a *function* in short, is a mapping $f : \{0,1\}^n \rightarrow \{0,1\}$. $T(f) = \{v \mid f(v) = 1\}$ and $F(f) = \{v \mid f(v) = 0\}$ are set of the true vectors and false vectors of f, respectively. A function f is *positive* (also called *monotone*) if $v \leq w$ implies $f(v) \leq f(w)$. A *Horn* function f has a well-known algebraic characterization, given by

$$f(v \wedge w) \leq f(v) \wedge f(w),$$

which is equivalent to the condition that $F(f)$ is closed under intersection. Equivalent definitions of positive and Horn functions can be given in terms of *disjunctive normal form* (DNF). A DNF φ is a *disjunction* $\bigvee_{i=1}^{k} t_i$ of terms. The length of a DNF (or arbitrary formula) φ, denoted by $|\varphi|$, is the number of symbols in φ. A DNF $\varphi = \bigvee_i t_i$ is called *positive* if all t_i are positive, and *Horn* if all t_i are Horn. We call a function *positive* (resp., *Horn*) if and only if it can be represented by some positive (resp., Horn) DNF.

In this paper, we are interested in restricting Horn functions further by imposing semantical balanced conditions on the true and false vectors. A function f is called *bidual Horn* if $T(f)$ is closed under union and $F(f)$ is closed under intersection. The class of all these functions is denoted by $\mathcal{C}_{BH}$. Note that f is bidual Horn if and only if f and f^d are Horn, where $f^d(x) = \overline{f}(\overline{x})$. For example, $f = \overline{x}_1x_2x_3 \vee x_1\overline{x}_3x_4 \vee x_2x_3x_4$ is bidual Horn, because $f^d = (\overline{x}_1 \vee x_2 \vee x_3)(x_1 \vee \overline{x}_3 \vee x_4)(x_2 \vee x_3 \vee x_4) = \overline{x}_1x_4 \vee x_1x_2 \vee x_2\overline{x}_3 \vee x_2x_4 \vee x_1x_3 \vee x_3x_4$. In particular, every positive function is bidual Horn.

3 Recognition of bidual Horn functions

We introduce some additional notations. For a term t, define the sets of indices $P(t)$ and $N(t)$ by $t = \bigwedge_{j \in P(t)} x_j \bigwedge_{j \in N(t)} \bar{x}_j$, and let $V(t) = P(t) \cup N(t)$. Furthermore, for a pair of terms t_i and t_j, let us denote by $t^+_{i,j}$ the positive term such that $P(t^+_{i,j}) = P(t_i) \cup P(t_j)$, and by $t^\pm_{i,j}$ the positive term such that $P(t^\pm_{i,j}) = V(t_i) \cup V(t_j)$. For example, if $t_1 = x_1\overline{x}_2x_3$ and $t_2 = \overline{x}_1x_4$, then $t^+_{1,2} = x_1x_3x_4$ and $t^\pm_{1,2} = x_1x_2x_3 x_4$.

Lemma 1 *Let φ be a Horn DNF. Then φ represents a bidual Horn function if and only if*

$$t^+_{i,j} \leq \varphi \qquad (\textit{equivalently, } t^\pm_{i,j} \leq \varphi) \qquad (1)$$

holds for all pairs of Horn terms t_i and t_j in φ such that $|N(t_i) \cup N(t_j)| = 2$. □

Exploiting this lemma, we obtain a polynomial time algorithm for checking the biduality of a Horn DNF.

Theorem 1 *Given a Horn DNF φ, deciding whether it represents a bidual Horn function can be done in $O(m^2|\varphi|)$ time, where m denotes the number of terms in φ.* □

However, we can show that recognition of the biduality from a non-Horn formula is difficult in general.

Theorem 2 *Let φ be a formula. Then deciding whether φ represents a bidual function is* co-NP*-complete, even if φ is a DNF.* □

4 Shortest DNFs of a bidual Horn function

In this section, we show that term-shortest and literal-shortest DNFs can be computed in polynomial time from a given Horn DNF. These contrast with the known negative results that computing a term-shortest DNF or a literal-shortest DNF of an arbitrary Horn DNF is NP-hard. Thus, bidual Horn functions constitute an important subclass of Horn functions for which computing a literal-shortest and term-shortest prime DNFs is polynomial.

Theorem 3 *Let φ is a Horn DNF of a bidual Horn function f. Then a term-shortest prime DNF of f can be computed from φ in $O(|\varphi|^2)$ time.* □

Theorem 4 *Let φ is a Horn DNF of a bidual Horn function f. Then a literal-shortest DNF of f can be computed in $O(|\varphi|(m_h^2 m_p + |\varphi|))$ time, where m_h and m_p denote the numbers of Horn and positive terms in φ, respectively.* □

References

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