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Duality for an Optimal Stopping Problem in Dynamic Fuzzy Systems

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We, in [1,3], defined a dynamic fuzzy system and studied its potential theory. Further, in [2], we analysed the dynamic fuzzy system, using the path-structure. This paper deals with a dualilty for an optimal stopping problem in [2].

1. Dynamic fuzzy systems

Let S be a metric space. We write a fuzzy set on S by its membership function $\tilde{s} : S \mapsto [0,1]$. For a fuzzy set $\tilde{s}$ on S, its α-cut $\tilde{s}_\alpha$ is defined by

$$\tilde{s}_\alpha := \{x \in S \mid \tilde{s}(x) \geq \alpha\} \ (\alpha \in (0,1]) \quad \text{and} \quad \tilde{s}_0 := \mathrm{cl}\{x \in S \mid \tilde{s}(x) > 0\},$$

where cl denotes the closure of a set. $\mathcal{E}(S)$ denotes the set of all countable unions of closed subsets of S, so called F_σ-sets. $\mathcal{F}(S)$ is the set of all fuzzy sets $\tilde{s}$ on S satisfying the following conditions (i) and (ii):

(i) $\tilde{s}_\alpha \in \mathcal{E}(S)$ for $\alpha \in [0,1]$;

(ii) $\bigcap_{\alpha' < \alpha} \tilde{s}_{\alpha'} = \tilde{s}_\alpha$ for $\alpha \in (0,1]$.

And, $\mathcal{G}(S)$ is the set of all fuzzy sets $\tilde{s}$ on S for which there exists a sequence $\{\tilde{s}_n\}_{n=0}^\infty \subset \mathcal{F}(S)$ such that $\sup_{n \geq 0} \tilde{s}_n(x) = \tilde{s}(x)$, $x \in S$. We use operations $\bigwedge$ and $\bigvee$ for real numbers in the sense of the maximum/supremum and the minimum/infremum respectively.

Let a time space $\mathbf{N} := \{0,1,2,3,\cdots\}$, and let a state space E be a finite-dimensional Euclidean space. We put a path space by $\Omega := \prod_{k=0}^\infty E$ and we represent a path by $\omega = (\omega(0),\omega(1),\omega(2),\cdots) \in \Omega$. Define a map $X_n(\omega) := \omega(n)$ for $n \in \mathbf{N}$ and $\omega = (\omega(0),\omega(1),\omega(2),\cdots) \in \Omega$. We define σ-fields $\mathcal{M}_n := \sigma(X_0, X_1, \cdots, X_n)$ [1] for $n \in \mathbf{N}$ and $\mathcal{M} := \sigma(\bigcup_{n\in\mathbf{N}} \mathcal{M}_n)$ [2]. In this paper, $X := \{X_n\}_{n \in \mathbf{N}}$ is called a dynamic fuzzy system, and the law of the transition is defined as follows. Let $\tilde{q}$ be time-invariant upper semicontinuous fuzzy relations on $E \times E$ satisfying the normality condition: $\sup_{x\in E} \tilde{q}(x,y) = 1$ $(y \in E)$ and $\sup_{y \in E} \tilde{q}(x,y) = 1$ $(x \in E)$. Then $\tilde{q}$ means a transition fuzzy relation. Define $\tilde{P}(\Lambda) := \sup_{\omega\in\Lambda} \bigwedge_{n\in\mathbf{N}} \tilde{q}(X_n(\omega), X_{n+1}(\omega))$ for $\Lambda \in \mathcal{M}$. We define a fuzzy expectation by the possibility measure $\tilde{P}$: For an initial state $x \in E$ and $\mathcal{M}$-measurable fuzzy sets $h \in \mathcal{F}(\Omega)$,

$$E_x(h) := \int_{\{\omega\in\Omega:\omega(0)=x\}} h(\omega)\, \mathrm{d}\tilde{P}(\omega),$$

where $\int \mathrm{d}\tilde{P}$ is Sugeno integral.

We call a map $\tau : \Omega \mapsto \mathbf{N} \cup \{\infty\}$ an $\mathcal{E}$-stopping time if $\{\tau = n\} \in \mathcal{M}_n \cap \mathcal{E}(\Omega)$ for all $n \in \mathbf{N}$. In this paper, we call maps from $\mathcal{G}(E)$ to $\mathcal{G}(E)$ fuzzy transition operators. For an $\mathcal{E}$-stopping time τ, we define a fuzzy transition operator $P_\tau : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ by

$$P_\tau \tilde{s}(x) := E_x(\tilde{s}(X_\tau)) \quad x \in E \quad \text{for } \tilde{s} \in \mathcal{G}(E),$$

[1] It denotes the smallest σ-field on Ω relative to which $X_0, X_1, \cdots, X_n$ are measurable.
[2] It denotes the smallest σ-field generated by $\bigcup_{n\in\mathbf{N}} \mathcal{M}_n$.

where $X_\tau := X_n$ on $\{\tau = n\}$, $n \in \mathbf{N} \cup \{\infty\}$. Clearly, P_0 is an identity operator and $P := P_1$ is given by

$$P\tilde{s}(x) = \sup_{y \in E} \{\tilde{q}(x,y) \wedge \tilde{s}(y)\} \quad x \in E \quad \text{for } \tilde{s} \in \mathcal{G}(E). \tag{1.1}$$

2. Duality for an Optimal Stopping Problem

A map $\mu : \mathcal{E}(E) \mapsto [0,1]$ is called a possibility measure on E if it satisfies (i) — (iii):

(i) $\mu(\phi) = 0$ and $\mu(E) = 1$;

(ii) If $A, B \in \mathcal{E}(E)$ satisfy $A \subset B$, then $\mu(A) \le \mu(B)$;

(iii) If $\{A_n\}_{n \in \mathbf{N}} \subset \mathcal{E}(E)$, then $\mu(\bigcup_{n \in \mathbf{N}} A_n) = \sup_{n \in \mathbf{N}} \mu(A_n)$.

We define a fuzzy integration of a fuzzy set $\tilde{s} \in \mathcal{G}(E)$ by the possibility measure μ:

$$\mu(\tilde{s}) := \int\!\!\!\!-_E \tilde{s}(x)\ \mathrm{d}\mu(x) = \sup_{\alpha \in [0,1]} (\alpha \wedge \mu(\tilde{s}_\alpha)).$$

Let $\mathcal{P}$ be the set of all possibility measures on E. We introduce a partial order on $\mathcal{P}$ as follows: For $\mu, \nu \in \mathcal{P}$, $\mu \ge \nu$ means that $\mu(A) \ge \nu(A)$ for all $A \in \mathcal{E}(E)$. We define $\mu L(\tilde{s}) := \mu(L\tilde{s})$ for $\tilde{s} \in \mathcal{G}(E)$ and fuzzy transition operators $L : \mathcal{G}(E) \mapsto \mathcal{G}(E)$. A fuzzy set $\tilde{s} \in \mathcal{G}(E)$ is called P-superharmonic if $\tilde{s}(x) \ge P\tilde{s}(x)$ for all $x \in E$. And, a possibility measure $\mu \in \mathcal{P}$ is also called P-excessive if $\mu \ge \mu P$.

We give a duality for Snell's optimal stopping problem in [2]. Let $\mu \in \mathcal{P}$ be a fixed initial possibility measure. Let $\tilde{s} \in \mathcal{G}(E)$ be a fuzzy goal. We consider a problem:

Maximize $\mu P_\tau(\tilde{s})$ with respect to finite $\mathcal{E}$-stopping times τ.

Then, we define a possibility measure

$$\nu(\tilde{s}) := \sup_{\tau : \text{finite } \mathcal{E}-\text{stopping times}} \mu P_\tau(\tilde{s}), \quad \tilde{s} \in \mathcal{G}(E). \tag{2.1}$$

Theorem 2.1 (Duality in Snell's optimal stopping problem). *ν is the smallest P-excessive possibility measure dominating μ. Further, it holds that*

$$\nu(\tilde{s}) = \min_{\mu' \in \mathcal{P} : \mu' \ge \mu,\, \mu' \le \mu' P} \mu'(\tilde{s}) = \min_{\tilde{r} : P-\text{superharmonic}, \tilde{r} \ge \tilde{s}} \mu(\tilde{r}) \quad \textit{for } \tilde{s} \in \mathcal{G}(E). \tag{2.2}$$

References

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