1996年度日本オペレーションズ・リサーチ学会
秋季研究発表会

Mark-Choose-Cut Algorithms for Fair and Strongly Fair Division

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1 Introduction

The problem of cake division is to divide a cake among a certain number of individuals in such a way that each individual is satisfied.

We assume that there are n individuals, and n associated finitely additive, non-atomic measures $m_1, m_2, \cdots, m_n$, each defined on some algebra of subsets of cake $\mathcal{C}$, such that for each m_i, $m_i(\mathcal{C}) = 1$. An *entitlement sequence* $\langle \alpha_1, \alpha_2, \cdots, \alpha_n \rangle$ is a sequence of positive numbers satisfying that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. The partition $\langle \mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n \rangle$ of $\mathcal{C}$ is *fair* with respect to the sequence of measures $\langle m_1, m_2, \cdots, m_n \rangle$ and the entitlement sequence $\langle \alpha_1, \alpha_2, \cdots, \alpha_n \rangle$ if and only if for each $i = 1, 2, \cdots, n$, $m_i(\mathcal{P}_i) \geq \alpha_i$. Furthermore, the partition is *strongly fair* if and only if $m_i(\mathcal{P}_i) > \alpha_i$ holds for all i. See Barbanel, J. B. (1995).

Barbanel (1995) gives game-theoretic algorithms for fair and strongly fair cake division with entitlements. However, the number of pieces resulting from his algorithm is not bounded even for the case of two individuals. In other words, the number of pieces could be made arbitrarily large by a suitable choice of the measures corresponding to the individuals' evaluations. However, in a land division problem, a parcel of land is valuable only if it is sufficiently large. In contrast, an isolated tiny piece of land is valueless. In other words, in order to deal with land division problems, it is important to find an algorithm so that the number of resulting pieces is as small as possible. Such an algorithm is theoretically important, see Brams, Taylor and Zwicker (1995) (which gives moving-knife algorithms for envy-free division so that the number of cuts is as small as possible). The purpose of this paper is to provide such algorithms.

2 Basic Model

We only allow that our knife parallel cuts the cake and suppose that one cut only divides the original continuous piece into 2 continuous pieces. Therefore we actually treat the cake as if it is of one-dimension. The whole cake is supposed to be continuous, and has two end points E_1 and E_2. We temporarily put the two ends together and denote the common point as S, remembering that when we finally count the number of resulting pieces of partition, we must remember that the original cake $\mathcal{C}$ is not continuous at S. Then the cake is now regarded as a circle with center O. Any point $A \in \mathcal{C}$ has a one-to-one correspondence with a counterclockwise rotation of radius from OS to OA. Given a positive integer k, we use $k \circ \mathcal{C}$ to refer $\mathcal{C}$ when the domain of $\angle SOA$ is extended from $(0, 2\pi]$ to $(0, 2k\pi]$. Therefore any $\theta \in (0, 2k\pi]$ has a one-to-one correspondence with a point A_θ in the circle $k \circ \mathcal{C}$, which satisfies $\angle SOA_\theta = \theta$. In this way, an arc AB of $k \circ \mathcal{C}$ also determines a central angle $\angle AOB = \angle SOB - \angle SOA$. We denote $S_i = A_{2i\pi}$ for $i = 1, 2, \cdots, k$. Each S_i coincides with S physically.

Throughout the paper, any measure is finitely additive, non-atomic, defined on some algebra of subsets of $\mathcal{C}$, which gives measure 1 to $\mathcal{C}$.

3 Fair Cake Division

3.1 The case of two players

This subsection considers the case that $n = 2$. We write $\alpha = \alpha_1$, and therefore $\alpha_2 = 1 - \alpha$.

We let one player choose a suitable k, and the other player mark a division of $(0, 2k\pi]$ suitably, so that one player is satisfactory with one arc corresponding to a resulting central angle and the other player is satisfactory with the remains. The algorithm is as follows.

Algorithm 3.1:

1. *Player* 2 *marks a central angle* θ_1 *which corresponds to an arc* $\widehat{SA_{\theta_1}} = \mathcal{Q}_1$.
   Player 2's strategy: Mark the angle so that $m_2(\mathcal{Q}_1) = \alpha$.

2. *Player* 1 *either says 'cut' or announces a positive integer* l *and let* $k = [l\alpha]$. *If Player* 1 *says 'cut', then the cake is divided into* $\mathcal{P}_1 = \mathcal{Q}_1$

and $\mathcal{P}_2 = \mathcal{C} - \mathcal{Q}_1$, and the game is over.
Player 1's strategy: If $m_1(\mathcal{Q}_1) \geq \alpha$, then Player 1 says 'cut'. If $m_1(\mathcal{Q}_1) < \alpha$, then Player 1 chooses an l which satisfies

$$l\alpha - [l\alpha] < \alpha - m_1(\mathcal{Q}_1). \quad (3.1)$$

3. *Player 2 gives a partition of interval $(\theta_1, 2k\pi]$ into $l-1$ intervals $(\theta_1, \theta_2]$, $(\theta_2, \theta_3]$, $\cdots$, $(\theta_{l-1}, \theta_l] = (\theta_{l-1}, 2k\pi]$. In other words, player 2 marks arcs $\mathcal{Q}_2 = \widehat{A_{\theta_1} A_{\theta_2}}$, $\mathcal{Q}_3 = \widehat{A_{\theta_2} A_{\theta_3}}$, $\cdots$, $\mathcal{Q}_l = \widehat{A_{\theta_{l-1}} S_k}$ consecutively and circularly.*
Player 2's strategy: Mark the arcs so that their measures are the same.

4. *Player 1 chooses one of $\mathcal{Q}_i$ $(i = 2, 3, \cdots, l)$, which is supposed to be $\mathcal{Q}_j$.*
Player 1's strategy: Choose one which Player 1 measures as the largest.

5. *The arc $\mathcal{Q}_j$ is cut from $\mathcal{C}$ and the cake is divided into $\mathcal{P}_1 = \mathcal{Q}_j$ and $\mathcal{P}_2 = \mathcal{C} - \mathcal{Q}_j$. The game is over.*

It is easy to see that 3 is the least number to form a fair division.

3.2 The case for n players

For a general n, we have

Theorem 1 *There is a game-theoretic algorithm that produces a partition $\langle \mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n \rangle$ of $\mathcal{C}$ which is fair with respect to any measures $m_1, m_2, \cdots, m_n$ and any entitlement sequence $\langle \alpha_1, \alpha_2, \cdots, \alpha_n \rangle$. Furthermore, the algorithm results in at most $2 \times 3^{n-2} + 1$ pieces.*

Algorithm 3.2:

1. *Let Player 1, Player 2, $\cdots$, Player k divide $\mathcal{C}$ to form a partition $\langle \mathcal{Q}_1, \mathcal{Q}_2, \cdots, \mathcal{Q}_k \rangle$, which is fair with respect to $m_1, m_2, \cdots, m_k$ and entitlement sequence*

$$\langle \frac{\alpha_1}{1-\alpha_{k+1}}, \frac{\alpha_2}{1-\alpha_{k+1}}, \cdots, \frac{\alpha_k}{1-\alpha_{k+1}} \rangle.$$

2. *By use of Algorithm 3.1, for $i = 1, 2, \cdots, k$, we let Player $k+1$ divide $\mathcal{Q}_i$ with Player i successively, to form a partition $\langle \mathcal{Q}_i', \mathcal{P}_i \rangle$, which is fair with respect to measures m_{k+1}' and m_i', entitlement sequence $\langle \alpha_{k+1}, 1-\alpha_{k+1} \rangle$, where*

$$m_{k+1}'(\cdot) = \frac{m_{k+1}(\cdot)}{m_{k+1}(\mathcal{Q}_i)}, \quad m_i'(\cdot) = \frac{m_i(\cdot)}{m_i(\mathcal{Q}_i)}.$$

3. *Output partition $\langle \mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_k, \mathcal{P}_{k+1} = \cup_{i=1}^{k} \mathcal{Q}_i' \rangle$, and the game is over.*

4 Strongly Fair Cake Division

As an evidence of the existence of strongly fair division, suppose that we are given a piece $\mathcal{D}$ of the cake $\mathcal{C}$ such that $m_i(\mathcal{D}) = \gamma > m_j(\mathcal{D}) = \delta$ (where γ, $\delta \in (0,1)$), for some players i and j. Since all measures are atomless, without loss of generality, we assume that $\mathcal{D}$ is a continuous piece. Otherwise, we can use a suitable continuous subset of $\mathcal{D}$ to replace $\mathcal{D}$. For convenience, we denote $i = 1$, $j = 2$, and $\overline{\mathcal{D}} = \mathcal{C} - \mathcal{D}$.

Theorem 2 *There is a game-theoretic algorithm that produces a partition $\langle \mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n \rangle$ of $\mathcal{C}$ which is strongly fair with respect to any measures $m_1, m_2, \cdots, m_n$ and any entitlement sequence $\langle \alpha_1, \alpha_2, \cdots, \alpha_n \rangle$. Furthermore, the algorithm results in at most $4 \times 3^{n-2} + 1$ pieces.*

5 Conclusion

1. We stated in Section 3.1, that the number of pieces $2 \times 3^{n-2} + 1$ is the least in the case of $n = 2$. However, we don't know whether it is true for a general n. This problem is left for future research.

2. In the case that $n = 2$, we can actually divide the cake into exactly two continuous pieces in many application cases, if we lift the restriction of parallel cutting the cake.

3. The above result can be actually generalized to the case of strongly fair division, if $\alpha > \delta$ and $\mathcal{D}$ is contained in the interior of $\mathcal{C}$.

References

Barbanel, J. B. (1995). "Game-theoretic algorithms for fair and strongly fair cake division with entitlements," *Colloquium Mathematicum* 69, 59-73.

Brams, S. J., Taylor, A. D. and Zwicker W. S. (1995). "A moving-knife solution to the four-person envy-free cake division problem," *Proceedings of the American Mathematical Society*, forthcoming.