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AN EXPLICIT SOLUTION FOR AN M/GI/1/N QUEUE WITH VACATION TIME AND EXHAUSTIVE SERVICE DISCIPLINE

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1. Introduction

The time division multiple access (TDMA) scheme is practical in the areas of communications. Communication engineers frequently encounter a teletraffic issue how to design a buffer capacity in the TDMA environment (see Stuck and Arthurs [4]). The issue then necessiates a single-server finite capacity queue with *vacation time* and *exhaustive* service discipline.
Assuming Poisson input Lee [3] already provided a numerical algorithm for this system via the standard embedded Markov chain technique. As the embedded points, he took the service completion epochs and server vacation completion epochs. To obtain the queue length distribution at an arbitrary time, he applied the supplementary variable technique and the sample biasing technique.
Here, we treat the same queueing system as in Lee [3] but present a simpler analysis than Lee's. We show that service completion epochs are enough for the queue length to form an embedded Markov chain. This simpler analysis enables us to find an explicit solution for the steady-state queue length distribution. To the best of the authors' knowledge, there are no results on the explicit solution for finite capacity queues.

2. The model

We consider an M/GI/1/N queue, where the input process is Poissonian with rate λ, the service times form a sequence of i.i.d. random variables with distribution function $S(x)$ and N equals the number of waiting places in the queue, including the space for the customer that may be in service. We assume that accepted customers by the system are served by a single server exhaustively, where a customer is accepted by the system if the number of customers in the system is less than N. Whenever the queue becomes empty the server starts a vacation with distribution function $V(x)$. If the queue is still empty upon his return, he takes another independent vacation with distribution function $V(x)$. We assume furtheron that the service discipline is non-preemptive and the service order is FIFO.

3. The queue length distribution at a departure epoch

By π_j, $j = 0, \ldots, N-1$, we denote the steady state probability that j customers are left in the system at a departure epoch of a customer. It is easy to see, that π_j satisfy the following equations

$$c_k \pi_0 + \sum_{i=1}^{k+1} g_{k+1-i}\pi_i = 0, \qquad 0 \le k \le N-2 \tag{3.1}$$

and the normalization condition

$$\sum_{j=0}^{N-1} \pi_j = 1, \tag{3.2}$$

where

$$\begin{aligned}
g_k &= \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x}\, dS(x),\ k = 0,2,3,\ldots \\
g_1 &= \int_0^\infty (\lambda x) e^{-\lambda x}\, dS(x) - 1, \\
c_0 &= \varphi_1 g_0 - 1, \\
c_k &= \sum_{i=1}^{k+1} \varphi_i g_{k-i+1} + \varphi_k,\ 1 \le k \le N-2, \\
\varphi_j &= \sum_{l=0}^{\infty} (h_0)^l h_j = \frac{h_j}{1-h_0},\ 1 \le j \le N-1 \\
\varphi_N &= \sum_{l=0}^{\infty} (h_0)^l \sum_{j=N}^{\infty} h_j = \frac{\sum_{j=N}^{\infty} h_j}{1-h_0} \\
h_j &= \int_0^\infty \frac{(\lambda x)^j}{j!} e^{-\lambda x}\, dV(x),\ j = 0,1,\ldots
\end{aligned}$$

By induction on n one can solve the equations (3.1) and (3.2) explicitly.

Theorem 3.1 *The probabilities that there are n customers left at a departure epoch are explicitly obtained as*

$$\pi_0 = \frac{(-g_0)^{N-1}}{\sum_{j=0}^{N-1} (-g_0)^{N-j-1} \sum_{\delta \in B_j^{\leq N-2}} c_{\delta_1} g_{\delta_2} \cdots g_{\delta_j}}$$

$$\pi_n = \frac{\sum_{j=1}^{n} (-g_0)^{n-j} \sum_{\delta \in B_j^{=n-1}} c_{\delta_1} g_{\delta_2} \cdots g_{\delta_j}}{\sum_{j=0}^{N-1} (-g_0)^{n-j} \sum_{\delta \in B_j^{\leq N-2}} c_{\delta_1} g_{\delta_2} \cdots g_{\delta_j}}$$

$$1 \leq n \leq N-1,$$

where $B_j^{\leq k}$ ($B_j^{=k}$) is the set of all j-tuples $\delta = (\delta_1, \ldots, \delta_j)$ with $\delta_1 \in \mathbb{N}_0$, $\delta_i \in \mathbb{N}$ $(i = 2, \ldots, j)$, $\sum_{i=1}^{j} \delta_i \leq k$ ($\sum_{i=1}^{j} \delta_i = k$) and $\sum_{\delta \in B_0^{\leq k}} c_{\delta_0} = 1$ for each $k \geq j-1$.

Remark 3.1 *For a finite capacity queue without vacation we can obtain the explicit solution for the probabilities that there are n customers left at a departure epoch by setting*

$$\begin{aligned} c_0 &= g_0 - 1, \\ c_k &= g_k, \qquad k \geq 1. \end{aligned}$$

4. The queue length distribution at an arbitrary time in steady state

In this section we will derive the probabilities π_j^*, that there are j customers in the system at an arbitrary time in steady state $(j = 0, \ldots, N)$. Let ρ' be the probability that the server is busy, then by using the PASTA property (see Wolff [5]) and applying Little's law, we obtain the following lemma.

Lemma 4.1 *It holds that*

$$\rho' = \lambda(1 - \pi_N^*)\mathbf{E}(S), \tag{4.3}$$

where π_N^* is the probability that N customers are in the system, and $\mathbf{E}(S)$ is the expected service time.

By considering the point process which is formed by the beginning epochs of busy periods and the point process which is formed by the end epochs of busy periods, and noting that they have the same intensity, one can obatin the following theorem, which links π_N^* with π_0.

Theorem 4.2 *The probability that the server is busy (not on vacation) is given by*

$$\rho' = \frac{\mathbf{E}(S)(1-h_0)}{\mathbf{E}(V)\pi_0 + \mathbf{E}(S)(1-h_0)}, \tag{4.4}$$

where $\mathbf{E}(V)$ is the expected vacation time.

Because of the PASTA property, we see that π_j^* is also the probability that there are j customers in the system just before an arrival. Thus, the generalized version of Burke's theorem (see Burke [1] and Cooper [2]) is applied to get

$$\pi_j = \frac{\pi_j^*}{1 - \pi_N^*}, \qquad j = 0, \ldots, N-1.$$

Together with (4.3) and (4.4) we obtain the following theorem.

Theorem 4.3 *The queue length distribution $\{\pi_j^*; j = 0, \ldots, N\}$ at an arbitrary time in steady state is obtained as*

$$\begin{aligned} \pi_j^* &= \frac{\pi_j(1-h_0)\lambda^{-1}}{\mathbf{E}(V)\pi_0 + \mathbf{E}(S)(1-h_0)}, \\ &\qquad j = 0, \ldots, N-1 \end{aligned} \tag{4.5}$$

$$\pi_N^* = 1 - \frac{(1-h_0)\lambda^{-1}}{\mathbf{E}(V)\pi_0 + \mathbf{E}(S)(1-h_0)}.$$

Remark 4.2 *By using Therorem 3.1 we are now able to obtain an explicit solution for the queue length distribution at an arbitrary time in steady state.*

Remark 4.3 *Equations (4.5) are seen to coincide with equation (9) given in Lee [3]. Note that the calculation of $\{\pi_i;\ i = 0, \ldots, N-1\}$ given by (3.1) and (3.2) is simpler than the calculation of $\{p_i;\ i = 0, \ldots, N-1\}$ in Lee [3], and that the argument for deriving $\{\pi_j^*\}$ is fairly lengthy in Lee [3]. We obtained a more efficient way to calculate the probabilities $\{\pi_i^*;\ i = 0, \ldots, N\}$.*

References

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