BOUNDS OF CYCLE TIMES OF A U-SHAPED PRODUCTION LINE

Koichi Nakade*, Nagoya Institute of Technology
Katsuhisa Ohno, Nagoya Institute of Technology
J. George Shanthikumar, University of California, Berkeley

1. Introduction

In a Just-In-Time (JIT) production system, a U-shaped layout and multi-function workers have been introduced to achieve a *single-unit production and conveyance* ("Ikko-Nagashi," in Japanese) at a low production cost. Recently, the cycle times and the waiting times in a U-shaped line with multi-function workers have been analyzed in some special cases([1] – [3]). In this report, we derive the upper and lower bounds of the expected waiting times and cycle times of the U-shaped production line with a single multi-function worker where processing times of items, operation times, and walking times of the worker between machines are sequences of mutually independent and identically distributed random variables.

2. A U-shaped Production Line with a Multi-Function Worker

We consider a U-shaped production line with a single multi-function worker. The worker handles machines 1 through K. The facility has enough raw material in front of machine 1. The material is processed at machines 1 through K sequentially, and departs from the line as a finished product. Let $\hat{K} = \{1, \ldots, K\}$. When the worker arrives at machine $k \in \hat{K}$ and finds the processing of the preceding item completed, he removes it from machine $k \in \hat{K}$, sends it to machine $k+1$, attaches the present item to machine k and switches it on. After this operation on machine k, he walks to machine $k+1$. If the preceding item is still in process at his arrival, then he waits for the end of its processing before operation. It is assumed as an initial condition that at time 0, one item has been already processed at each machine. That is, in the first cycle the worker does not wait at all machines. We use the following notations: for $k \in \hat{K}$ and $n \in \mathcal{Z} = \{1, 2, \ldots\}$,

$I_k(n)$: the nth processing time at machine k,

$S_k(n)$: the nth operation time of the worker on machine k,

$R_k(n)$: the nth walking time from machine k to machine $k+1$ (K to 1, if $k = K$),

$C(n)$: the nth cycle time, which is the time interval between the nth and the $(n+1)$st arrivals of the worker at machine 1, $[a]^+ = \max\{0, a\}$,

$\mathcal{R}_+ = [0, \infty)$.

We assume that $\{I_k(n); n \in \mathcal{Z}\}$, $\{S_k(n); n \in \mathcal{Z}\}$ and $\{R_k(n); n \in \mathcal{Z}\}$ for $k \in \hat{K}$ are sequences of independent and identically distributed random variables, and they are mutually independent.

3. Upper and Lower Bounds of Expected Waiting Times

We denote the waiting time of the worker at machine k in the nth cycle by $W_k(n)$. Then we have the following recursive equations with respect to $W_k(n)$[3]:

$$W_k(1) = 0 \text{ for } k \in \hat{K},$$

$$W_k(n) = [Y_k(n) - \textstyle\sum_{j<k} W_j(n) - \sum_{j>k} W_j(n-1)]^+$$

and

$$\sum_{j=1}^{k} W_j(n) = \max_{1 \le j \le k} [Y_j(n) - \sum_{m>j} W_m(n-1)]^+, \quad (1)$$

for $k \in \hat{K}$ and $n \ge 2$, where

$$Y_k(n) = I_k(n-1) - R_k(n-1) - \textstyle\sum_{j>k}(S_j(n-1) + R_j(n-1)) - \sum_{j<k}(S_j(n) + R_j(n)).$$

Theorem 1

For $n \ge 2$, it holds that

$$\sum_{k \in \hat{K}} E[W_k(n)] \le E[\max_{k \in \hat{K}} [Y_k(2)]^+]$$

and

$$E[\max_{k \in \hat{K}} [Y_k(2)]^+] \le \sum_{k \in \hat{K}} E[W_k(n-1) + W_k(n)]. \blacksquare$$

Let the expected values of $I_k(n)$, $S_k(n)$ and $R_k(n)$ be denoted by i_k, s_k and r_k for each $k \in \hat{K}$, respectively.

4. Computations of Upper Bounds

The upper bound $E[\max_{k \in \hat{K}} [Y_k(n)]^+]$ shown in Section 3 is not easy to compute, because $Y_1(n), \ldots, Y_K(n)$ are mutually dependent. We give the easily computable upper bounds in the following. We define $Z_k(n)$ as

$$Z_k(n) = R_k(n-1) + \textstyle\sum_{j>k}(S_j(n-1) + R_j(n-1)) + \displaystyle\sum_{j<k}(S_j(n) + R_j(n)),$$

and then

$$Y_k(n) = I_k(n-1) - Z_k(n) \text{ for } k \in \hat{K} \text{ and } n \in \mathcal{Z}.$$

Since $Z(n) = (Z_1(n), \ldots, Z_K(n))$ is weakly conditionally increasing in sequence (WCIS), they are associated(see [4]). Using this property we have

$$P(Y_k(n) \le y \text{ for all } k \in \hat{K}) \ge \prod_{k \in \hat{K}} P(Y_k(n) \le y),$$

for all $y \in \mathcal{R}_+$. Therefore, it follows that

$$E\left[\max_{k \in \hat{K}} [Y_k(n)]^+\right] \le E\left[\max_{k \in \hat{K}} [\tilde{Y}_k]^+\right],$$

where $\tilde{Y}_k$ has the same distribution as $Y_k(n)$ for $k \in \hat{K}$ and $\tilde{Y}_1, \ldots, \tilde{Y}_K$ are mutually independent.

5. A Lower Bound of the Expected Cycle Time

In this section, we derive another lower bound of the expected cycle time. We use the following notations: For each $k \in \hat{K}$ and $n \in \mathcal{Z}$,
$U_k(n)$: the time interval from the nth arrival to the nth departure of the worker at machine k,
$B_k(n)$: the time interval from the nth departure to the $(n+1)$st arrival of the worker at machine k.
It is clear that for each $k \in \hat{K}$,

$$U_k(n) = W_k(n)+S_k(n) = [I_k(n-1)-B_k(n-1)]^+ + S_k(n), \tag{2}$$

and

$$B_k(n) = R_k(n) + \sum_{j>k}(U_j(n) + R_j(n)) + \sum_{j<k}(U_j(n+1) + R_j(n+1)).$$

Then we have

$$C(n) = B_1(n) + \sum_{j=2}^{K} B_j(n-1) - \sum_{j=3}^{K}(j-2)U_j(n-1) - \sum_{j=2}^{K}(j-1)R_j(n-1) - (K-2)U_1(n) - \sum_{j=2}^{K-1}(K-j)U_j(n) - \sum_{j=1}^{K-1}(K-j)R_j(n) \tag{3}$$

We assume that the values of $E[U_k(n)]$, $E[B_k(n)]$ and $E[C(n)]$ converge to u_k^*, b_k^* and C^* as $n \to \infty$, respectively. In fact, it is shown in [2] that if $P(Y_k(2) \le 0 \text{ for all } k \in \hat{K}) > 0$ then $w_k^* = \lim_{n\to\infty} E[W_k(n)]$ exists for each $k \in \hat{K}$, which implies that there exist u_k^*, b_k^* and C^* by (2). Then from equation (3),

$$C^* = \sum_{j\in\hat{K}} b_j^* - (K-1)\sum_{j\in\hat{K}} r_j - (K-2)\sum_{j\in\hat{K}} u_j^*.$$

It also holds that

$$C^* = u_k^* + b_k^* \quad \text{for all } k \in \hat{K}. \tag{4}$$

From these equations, (u_k^*, b_k^*) is the solution of the following simultaneous equations: for $k \in \hat{K}$,

$$u_k + b_k = \sum_{j\in\hat{K}} b_j - (K-1)\sum_{j\in\hat{K}} r_j - (K-2)\sum_{j\in\hat{K}} u_j.$$

We can solve it to obtain u_k as

$$u_k = -\frac{K-2}{K-1}b_k + \frac{1}{K-1}\sum_{j\ne k} b_j - \frac{1}{K-1}\sum_{j\in\hat{K}} r_j, \quad \text{for } k \in \hat{K}. \tag{5}$$

Denote the right hand side of (5) by $\bar{u}_k(b)$, where $b = (b_1, \ldots, b_K)$. Let $u^* = (u_1^*, \ldots, u_K^*)$ and $b^* = (b_1^*, \ldots, b_K^*)$. Equation (5) implies that

$$u_k^* = \bar{u}_k(b^*), \quad \text{for } k \in \hat{K}. \tag{6}$$

Define $\hat{u}_k(x) = E[S_k(n)] + E[[I_k(n-1) - x]^+]$ for $x \in \mathcal{R}_+$. Since $I_k(n-1)$ and $B_k(n-1)$ are mutually independent and $\lim_{n\to\infty} B_k(n) \ge_{cx} b_k^*$, where $\ge_{cx}$ is a convex order (see [4], for example), from (2) it follows that for $k \in \hat{K}$

$$u_k^* = \lim_{n\to\infty} E[U_k(n)] = \lim_{n\to\infty} E[\hat{u}_k(B_k(n-1))] \ge \hat{u}_k(b_k^*).$$

We assume that there is a solution of the following equations, which is denoted by $\hat{b} = \{\hat{b}_k\}$: for $k \in \hat{K}$

$$\hat{u}_k(b_k) = \bar{u}_k(b), \tag{7}$$

If s_k and $F_k(x)$ are identical for all $k \in \hat{K}$, then $b_k = b_1$ for all $k \in \hat{K}$ and there is a solution of (7). We show another sufficient condition for the solution of (7) to exist. Note that equations (7) are equivalent to

$$g_{\tilde{k}}(b_{\tilde{k}}) = \frac{1}{K-1}\sum_{k\ne\tilde{k}} b_k, \tag{8}$$

$$g_k(b_k) + \frac{1}{K-1}b_k = g_{\tilde{k}}(b_{\tilde{k}}) + \frac{1}{K-1}b_{\tilde{k}} \quad k \ne \tilde{k}, \tag{9}$$

where $g_k(b) = \hat{u}_k(b) + \frac{K-2}{K-1}b + \frac{1}{K-1}\sum_{j\in\hat{K}} r_j$ and $\tilde{k} = \min\{k \in \hat{K}; s_k + i_k = \max_{j\in\hat{K}}(s_j + i_j)\}$.

Lemma 1 There is a solution $\{b_k\}$ of (9) for each fixed $b_{\tilde{k}} \ge \tilde{b}_{\tilde{k}}$, where $\tilde{b}_k = \inf\{x; F_k(x) > 0\}$. Moreover, if it holds that

$$g_{\tilde{k}}(\tilde{b}_{\tilde{k}}) \ge \frac{1}{K-1}\sum_{k\ne\tilde{k}} \tilde{b}_k, \tag{10}$$

where $\{\tilde{b}_k\}$ is a maximal solution of (9) when $b_{\tilde{k}} = \tilde{b}_{\tilde{k}}$, then there exists a unique solution $\{\hat{b}_k\}$ which satisfies (8) and (9) and hence satisfies (7). ■

Since for all $k \in \hat{K}$

$$u_k^* + b_k^* - (\bar{u}_k(\hat{b}) + \hat{b}_k) = \frac{1}{K-1}\sum_{j\in\hat{K}}(b_j^* - \hat{b}_j),$$

$\bar{u}_k(\hat{b}) + \hat{b}_k$ takes the same value for all $k \in \hat{K}$.

Theorem 2

$$C^* \ge \bar{u}_k(\hat{b}) + \hat{b}_k = \frac{1}{K-1}\sum_{j\in\hat{K}}(\hat{b}_j - r_j) = \hat{C},$$

for all $k \in \hat{K}$, that is, $\hat{C}$ is the lower bound of the cycle time C^*. ■

References

[1] K. Nakade and K. Ohno, *Proc. of Symposium on Performance Models for Info. Comm. Networks*, 404-413, 1995.

[2] K. Nakade and K. Ohno, *Queueing Systems*, 21, 183-197 (1995).

[3] K. Ohno and K. Nakade, to appear in *Journal of the Operations Research Society of Japan*, 1996.

[4] M. Shaked and J. G. Shanthikumar, *Stochastic Orders and Their Applications*, Academic Press, Boston, 1994.