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Mean Sojourn Times of Multiclass Feedback Queues with Gated Service Disciplines

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1 Introduction

We consider multiclass M/G/1 feedback queues with gated disciplines. Customers in the system are classified into J groups. Further, group i consists of L_i classes of customers. Let $\mathcal{S} \equiv \{(i,\alpha) : i = 1,\ldots,J \text{ and } \alpha = 1,\ldots,L_i\}$, and $J_c \equiv \sum_{i=1}^{J} L_i$. Class α customers belonging to group i ((i,α)-customers) arrive at station i from outside the system according to a Poisson process with rate $\lambda_{i\alpha}$ ($(i,\alpha) \in \mathcal{S}$). Service times $S_{i\alpha}$ of (i,α)-customers are independently and arbitrarily distributed with mean $E[S_{i\alpha}]$ and second moment $\overline{s^2}_{i\alpha}$ ($(i,\alpha) \in \mathcal{S}$). After receiving a service, an (i,α)-customer either feeds back to the system and is changed into a (j,β)-customer with probability $p_{i\alpha,j\beta}$, or departs from the system with probability $p_{i\alpha,00} = 1 - \sum_{j=1}^{J}\sum_{\beta=1}^{L_j} p_{i\alpha,j\beta}$ ($(i,\alpha),(j,\beta) \in \mathcal{S}$).

We consider *priority* scheduling algorithms for which group i has priority over group j if $i < j$. The system is separated into two parts which are called the 'service facility' and the 'waiting rooms' of the stations. The server selects one of the stations at a time, and then opens its gate, which separates the service facility from its waiting room. We consider the gated discipline, that is, when the gate opens, all customers at the station enters the service facility in batches at a time, and then the gate is immediately closed. The server serves customers in the service facility until he empties it, and then selects another station and opens its gate. Each arriving customer (exogenously or by feedback) joins one of the waiting rooms of the stations. The service order of customers in the service facility is FCFS.

We consider $\overline{T}_{i\alpha,j}$ and $\overline{T}_{i\alpha,j}(r)$ that satisfy equations: $\overline{T}_{i\alpha,j} = E[S_{i\alpha}] + \sum_{k=1}^{j}\sum_{\gamma=1}^{L_k} p_{i\alpha,k\gamma}\overline{T}_{k\gamma,j}$ and $\overline{T}_{i\alpha,j}(r) = r + \sum_{k=1}^{j}\sum_{\gamma=1}^{L_k} p_{i\alpha,k\gamma}\overline{T}_{k\gamma,j}$ ($0 \le j \le J, (i,\alpha) \in \mathcal{S}$ and $r \ge 0$). An empty sum, which often occurs at $j=0$, is defined to be 0 from now on. Further we define traffic intensities:

$$\rho_j^+ \equiv \sum_{i=1}^{j}\sum_{\alpha=1}^{L_i} \lambda_{i\alpha}\overline{T}_{i\alpha,j}, \qquad j = 0,1,\ldots,J. \tag{1.1}$$

Let (κ, a) denote the station-class pair of a customer being served, and let r be his remaining service time. Let $\mathcal{S}_0 \equiv \mathcal{S} \cup \{(0,0)\}$. Number of (i,α)-customers in the service facility (except for a customer being served) is denoted by $g_{i\alpha}$ ($(i,\alpha) \in \mathcal{S}$). Let $\mathbf{g}_i \equiv (g_{i\alpha} : \alpha = 1,\ldots,L_i)$, and $\mathbf{g} \equiv (\mathbf{g}_1,\ldots,\mathbf{g}_J)$. Number of (i,α)-customers in the waiting room is denoted by $n_{i\alpha}$ ($(i,\alpha) \in \mathcal{S}$). Let $\mathbf{n}_i \equiv (n_{i\alpha} : \alpha = 1,\ldots,L_i)$, and $\mathbf{n} \equiv (\mathbf{n}_1,\ldots,\mathbf{n}_J)$. These values at time t are denoted by the same notations followed by '(t)'.

The e^{th} customer (denoted by $\mathbf{c}^e$) arrives from outside the system at epoch σ_0^e ($e = 1,2,\ldots$). Let σ_k^e be a time epoch just when he arrives (by a feedback) at one of the stations after completing his k^{th} service stage ($k = 1,2,\ldots$). We denote informations of the system at time t by $L(t)$. Transition epochs of these processes consist of customer arrival epochs and service completion epochs. Then let $(X(t),\Gamma(t))$ denote a station-class pair of an arriving customer at the last transition epoch before t ($t \ge 0$). We define the stochastic process $\mathcal{Q} = \{\mathbf{Y}(t) = (X(t),\Gamma(t),\kappa(t),a(t),r(t),\mathbf{g}(t),\mathbf{n}(t),L(t)) : t \ge 0\}$ with the state space $\mathcal{E}$.

For $e = 1,2,\ldots$, $t \ge 0$ and $(i,\alpha) \in \mathcal{S}$, $C_{Wi\alpha}^e(t) \equiv 1$ if $\mathbf{c}^e$ stays in the waiting room as an (i,α)-customer at time t, or $C_{Wi\alpha}^e(t) \equiv 0$ otherwise. Further, $C_{Fi\alpha}^e(t) \equiv 1$ if $\mathbf{c}^e$ stays in the service facility or receives a service as an (i,α)-customer at time t, or $C_{Fi\alpha}^e(t) \equiv 0$ otherwise. Then we define

$$W_{i\alpha}(\mathbf{Y},e,l) \equiv E\left[\int_{\sigma_l^e}^{\infty} C_{Wi\alpha}^e(t)dt \middle| \mathbf{Y}(\sigma_l^e) = \mathbf{Y}\right], \tag{1.2}$$

$$W_{i\alpha}^I(\mathbf{Y},e,l) \equiv E\left[\int_{\sigma_l^e}^{\sigma_{l+1}^e} C_{Wi\alpha}^e(t)dt \middle| \mathbf{Y}(\sigma_l^e) = \mathbf{Y}\right], \tag{1.3}$$

$$F_{i\alpha}(\mathbf{Y},e,l) \equiv E\left[\int_{\sigma_l^e}^{\infty} C_{Fi\alpha}^e(t)dt \middle| \mathbf{Y}(\sigma_l^e) = \mathbf{Y}\right], \tag{1.4}$$

$$F_{i\alpha}^I(\mathbf{Y},e,l) \equiv E\left[\int_{\sigma_l^e}^{\sigma_{l+1}^e} C_{Fi\alpha}^e(t)dt \middle| \mathbf{Y}(\sigma_l^e) = \mathbf{Y}\right], \tag{1.5}$$

for $(i,\alpha) \in \mathcal{S}$, $l = 0,1,2,\ldots$, and $\mathbf{Y} \in \mathcal{E}$. It can be shown:

$$W_{i\alpha}(\mathbf{Y},e,l) = W_{i\alpha}^I(\mathbf{Y},e,l) + E[W_{i\alpha}(\mathbf{Y}(\sigma_{l+1}^e),e,l+1)|\mathbf{Y}], \tag{1.6}$$

$$F_{i\alpha}(\mathbf{Y},e,l) = F_{i\alpha}^I(\mathbf{Y},e,l) + E[F_{i\alpha}(\mathbf{Y}(\sigma_{l+1}^e),e,l+1)|\mathbf{Y}], \tag{1.7}$$

for $(i,\alpha) \in \mathcal{S}$, $l = 0,1,2,\ldots$ and $\mathbf{Y} \in \mathcal{E}$. (The above condition $\mathbf{Y}$ denote the following condition: $\mathbf{Y}(\sigma_l^e) = \mathbf{Y}$.)

2 Busy period processes

For any $\mathbf{Y} \in \mathcal{E}$ at any transition epoch, let $\mathcal{C} = \mathcal{C}(\mathbf{Y})$ be a set of some specified customers stay in the system at that time. For example, if an (i,α)-customer at the m^{th} position of the queue is in $\mathcal{C}$, then $(i,\alpha,m) \in \mathcal{C}$.

For any $\mathbf{Y} \in \mathcal{E}$ and any $\mathcal{C} = \mathcal{C}(\mathbf{Y})$, let $B^j(\mathbf{Y};\mathcal{C})$ be a (generalized) busy period of the system initiated with state $\mathbf{Y}$ until the first time when the system is cleared of the customers in $\mathcal{C}$ and customers at stations $1,\ldots,j$, except for customers who are initially in the system and are not in $\mathcal{C}$. Then we have

$$E[B^j(\mathbf{Y};\mathcal{C})] = \frac{\overline{T}_{\kappa a,j}(r) + \sum\sum_{(i,\alpha,m)\in\mathcal{C}^0}\overline{T}_{i\alpha,j}}{1-\rho_j^+},$$

where $C^0 = C \setminus \{\text{a customer being served}\}$ $(0 \le j \le J)$.

Let $\overline{N}^j_{l\delta}(\mathbf{Y};C)$ be the expected number of (l,δ)-customers at a completion epoch of $B^j(\mathbf{Y};C)$. Then

$$\overline{N}^j_{l\delta}(\mathbf{Y};C) = N_{l\delta}(C) + r\xi^j_{l\delta} + \chi^j_{\kappa a,l\delta} + \sum\sum_{(i,\alpha,m)\in C^0} (E[S_{i\alpha}]\xi^j_{l\delta} + \chi^j_{i\alpha,l\delta}),$$

for $0 \le j < l \le J$ and $(l,\delta) \in \mathcal{S}$, where $N_{l\delta}(C)$ is the number of (l,δ)-customers not in C. $\xi^j_{l\delta}$ and $\chi^j_{i\alpha,l\delta}$ are given in [1].

3 Initial sojourn times

For any $l = 0,1,2,\ldots$, let $\mathbf{Y} = (j,\beta,\kappa,a,r,\mathbf{g},\mathbf{n},L) \in \mathcal{E}$ be the state of the system at σ^e_l. Since $W^I_{i\alpha}(\mathbf{Y},e,l) = 0$ and $F^I_{i\alpha}(\mathbf{Y},e,l) = 0$ for $(i,\alpha) \ne (j,\beta)$, we consider the case: $(i,\alpha) = (j,\beta) \in \mathcal{S}$.

The length of time until the service period of $\mathbf{c}^e$ begins is equal to $B^{j-1}(\mathbf{Y};C^G)$ where $C^G \equiv C^G(\mathbf{Y})$ is a set composed of following customers: a customer being served currently, customers in the service facility and customers belonging to groups $1,\ldots,j-1$ who stay in the system at $\mathbf{c}^e$'s arrival epoch. Hence

$$\begin{aligned} & W^I_{j\beta}(\mathbf{Y},e,l) = E[B^{j-1}(\mathbf{Y},C^G)] \qquad (3.1)\\ = & \frac{\overline{T}_{\kappa a,j-1}(r) + \sum_{i=1}^{J}\sum_{\alpha=1}^{L_i} g_{i\alpha}\overline{T}_{i\alpha,j-1} + \sum_{i=1}^{j-1}\sum_{\alpha=1}^{L_i} n_{i\alpha}\overline{T}_{i\alpha,j-1}}{1-\rho^+_{j-1}}. \end{aligned}$$

The $\mathbf{c}^e$'s sojourn time $F^I_{j\beta}$ is a time to complete: 1) services of customers who are already in his group at his arrival epoch, and 2) his service. Then

$$F^I_{j\beta}(\mathbf{Y},e,l) = \sum_{\delta=1}^{L_j} n_{j\delta}E[S_{j\delta}] + E[S_{j\beta}]. \qquad (3.2)$$

Further, we have

$$E[n_{k\gamma}(\sigma^e_{l+1})|\mathbf{Y}] =$$
$$\begin{cases} \overline{N}^{j-1}_{k\gamma}(\mathbf{Y};C^G) + \lambda_{k\gamma}\left\{\sum_{\delta=1}^{L_j} n_{j\delta}E[S_{j\delta}] + E[S_{j\beta}]\right\} \\ \quad + \sum_{\delta=1}^{L_j} n_{j\delta}p_{j\delta,k\gamma}, & k \ne j, \gamma = 1,\ldots,L_k, \\ \lambda_{j\gamma}\left\{\sum_{\delta=1}^{L_j} n_{j\delta}E[S_{j\delta}] + E[S_{j\beta}]\right\} \\ \quad + \sum_{\delta=1}^{L_j} n_{j\delta}p_{j\delta,j\gamma}, & k = j, \gamma = 1,\ldots,L_j, \end{cases} \qquad (3.3)$$

$$E[g_{k\gamma}(\sigma^e_{l+1})|\mathbf{Y}] =$$
$$\begin{cases} 0, & k \ne j, \gamma = 1,\ldots,L_k, \\ \overline{N}^{j-1}_{j\gamma}(\mathbf{Y};C^G) - n_{j\gamma}, & k = j, \gamma = 1,\ldots,L_j. \end{cases} \qquad (3.4)$$

4 Expressions of the cost functions

We can find expressions of the cost functions $W_{i\alpha}(\cdot)$ and $F_{i\alpha}(\cdot)$ under some assumptions as follows.

For any $e = 1,2,\ldots$, define

$$\hat{W}_{i\alpha}(\mathbf{Y},e,l) \equiv \begin{cases} r\varphi_{i\alpha}(j,\beta,\kappa,a) + \\ (\mathbf{g},\mathbf{n})\mathbf{w}_{i\alpha}(j,\beta) + w_{i\alpha}(j,\beta,\kappa,a), & l = 0, \\ (\mathbf{g},\mathbf{n})\mathbf{w}_{i\alpha}(j,\beta) + w_{i\alpha}(j,\beta), & l > 0. \end{cases} \qquad (4.1)$$

$$\hat{F}_{i\alpha}(\mathbf{Y},e,l) \equiv \begin{cases} r\eta_{i\alpha}(j,\beta,\kappa,a) + \\ (\mathbf{g},\mathbf{n})\mathbf{f}_{i\alpha}(j,\beta) + f_{i\alpha}(j,\beta,\kappa,a), & l = 0, \\ (\mathbf{g},\mathbf{n})\mathbf{f}_{i\alpha}(j,\beta) + f_{i\alpha}(j,\beta), & l > 0. \end{cases} \qquad (4.2)$$

where $(i,\alpha) \in \mathcal{S}$ and $\mathbf{Y} = (j,\beta,\kappa,a,r,\mathbf{g},\mathbf{n},L) \in \mathcal{E}$.

Under some 'assumptions', we can show that the functions defined in (4.1) and (4.2) are solutions of equations (1.6) and (1.7), respectively.

We do not give here the precise definitions of the above coefficients $\varphi_{i\alpha}, \eta_{i\alpha}, \mathbf{w}_{i\alpha}, \mathbf{f}_{i\alpha}, w_{i\alpha}, f_{i\alpha}$. These constants can be explicitly determined by the coefficients of the functions $W^I_{i\alpha}$, $F^I_{i\alpha}$, $E[g_{k\gamma}(\sigma^e_{l+1})|\mathbf{Y}]$ and $E[n_{k\gamma}(\sigma^e_{l+1})|\mathbf{Y}]$. These discussions are similar to those in [1]. Uniqueness of the solutions of equations (1.6) and (1.7) may also be shown. Then $\hat{W}_{i\alpha}$ and $\hat{F}_{i\alpha}$ coincide with $W_{i\alpha}$ and $F_{i\alpha}$ defined by (1.2) and (1.4), respectively.

5 Steady state values

A set of arrival rates $\{\tilde{\theta}_{i\alpha} : (i,\alpha) \in \mathcal{S}\}$ is defined by a set of equations $\tilde{\theta}_{i\alpha} = \lambda_{i\alpha} + \sum_{j=1}^{J}\sum_{\beta=1}^{L_j} \tilde{\theta}_{j\beta}p_{j\beta,i\alpha}$ $((i,\alpha) \in \mathcal{S})$. Let $\tilde{r}^{\kappa a} = \tilde{\theta}_{\kappa a}\overline{s^2}_{\kappa a}/2$ $((\kappa,a) \in \mathcal{S})$ and $\tilde{r}^{00} = 0$. Further let $\tilde{q}^{\kappa a} = \tilde{\theta}_{\kappa a}E[S_{\kappa a}]$ $((\kappa,a) \in \mathcal{S})$ and $\tilde{q}^{00} = 1 - \rho^+_J$.

Under some steady state assumptions, we have

$$(\tilde{\mathbf{g}},\tilde{\mathbf{n}}) \equiv \lim_{t\to\infty}\frac{1}{t}\int_0^t (\mathbf{g}(u),\mathbf{n}(u))du = (\mathbf{s}-\tilde{\mathbf{s}})(\mathbf{I}-\mathbf{S})^{-1}, \qquad (5.1)$$

where

$$\tilde{\mathbf{s}} \equiv (\tilde{q}^{11},\ldots,\tilde{q}^{JL_J},0,\ldots,0) \in \mathcal{R}^{2J_c},$$

$$\mathbf{S} \equiv \sum_{j=1}^{J}\sum_{\beta=1}^{L_j} \lambda_{j\beta}(\mathbf{f}_{11}(j,\beta),\ldots,\mathbf{f}_{JL_J}(j,\beta), \mathbf{w}_{11}(j,\beta),\ldots,\mathbf{w}_{JL_J}(j,\beta)),$$

$$\mathbf{s} \equiv \sum_{j=1}^{J}\sum_{\beta=1}^{L_j} \lambda_{j\beta}(\tilde{f}_{11}(j,\beta),\ldots,\tilde{f}_{JL_J}(j,\beta), \tilde{w}_{11}(j,\beta),\ldots,\tilde{w}_{JL_J}(j,\beta)),$$

$$\tilde{w}_{i\alpha}(j,\beta) \equiv \sum\sum_{(\kappa,a)\in\mathcal{S}_0} \{\tilde{r}^{\kappa a}\varphi_{i\alpha}(j,\beta,\kappa,a) + \tilde{q}^{\kappa a}w_{i\alpha}(j,\beta,\kappa,a)\},$$

$$\tilde{f}_{i\alpha}(j,\beta) \equiv \sum\sum_{(\kappa,a)\in\mathcal{S}_0} \{\tilde{r}^{\kappa a}\eta_{i\alpha}(j,\beta,\kappa,a) + \tilde{q}^{\kappa a}f_{i\alpha}(j,\beta,\kappa,a)\}.$$

Let $\bar{W}_{i\alpha}(j,\beta)$ be the steady state average value of the *waiting times* of a customer spends in the waiting room as an (i,α)-customer, given that he has arrived from outside the system as a (j,β)-customer. Let $\bar{F}_{i\alpha}(j,\beta)$ be the steady state value of the *sojourn times* of the customer in the service facility $((i,\alpha),(j,\beta) \in \mathcal{S})$. Then

$$\bar{W}_{i\alpha}(j,\beta) = (\mathbf{s}-\tilde{\mathbf{s}})(\mathbf{I}-\mathbf{S})^{-1}\mathbf{w}_{i\alpha}(j,\beta) + \tilde{w}_{i\alpha}(j,\beta), \qquad (5.2)$$
$$\bar{F}_{i\alpha}(j,\beta) = (\mathbf{s}-\tilde{\mathbf{s}})(\mathbf{I}-\mathbf{S})^{-1}\mathbf{f}_{i\alpha}(j,\beta) + \tilde{f}_{i\alpha}(j,\beta). \qquad (5.3)$$

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