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Optimally Augmenting to Make a Biconnected Graph Four-Edge and Three-Vertex Connected

02004044 Kyoto University * ISHII Toshimasa
01403794 Kyoto University NAGAMOCHI Hiroshi
01001374 Kyoto University IBARAKI Toshihide

1 Introduction

Let $G = (V, E)$ stand for an undirected multigraph with a set V of *vertices* and a set E of *edges*. The connectivity augmentation problem has been extensively studied as an important problem in the network design problem.

The local edge-connectivity $\lambda_G(x, y)$ for two vertices $x, y \in V$ is defined to be the minimum size of a cut in G that separates x and y (i.e., x and y belong to different sides of X and $V - X$), or equivalently the maximum number of edge-disjoint path between x and y by Menger's theorem [1]. The local vertex-connectivity $\kappa_G(x, y)$ for two vertices $x, y \in V$ is defined to be the number of internally-disjoint paths between x and y in G. For a given integer k, we call G *k-edge-connected* (resp., *k-vertex-connected*) if $\lambda_G(x, y) \geq k$ (resp., $\kappa_G(x, y) \geq k$) holds for every $x, y \in V$. Given a multigraph $G = (V, E)$ and an integer k, the *edge-connectivity augmentation problem*, (resp., the *vertex-connectivity augmentation problem*) asks to augment G by adding the smallest number of new edges so that the resulting graph G' becomes k-edge-connected (resp., k-vertex-connected). Recently, many efficient algorithms are developed for solving the edge-connectivity augmentation problem and the vertex-connectivity augmentation problem.

In this paper, we consider the problem of augmenting the edge-connectivity and the vertex-connectivity of a given graph G simultaneously by adding the smallest number of new edges. For two given integers k and ℓ, we say that G is *(k, ℓ)-connected* if G is k-edge-connected and ℓ-vertex-connected. Given a multigraph $G = (V, E)$, and two integers k, ℓ, the *edge- and vertex-connectivity augmentation problem*, denoted by EVAP(k, ℓ), asks to augment G by adding the smallest number of new edges to G so that the resulting graph G' becomes (k, ℓ)-connected. Recently, it is shown in [2] that EVAP(k,2) can be solved in polynomial time for an integer k. In this paper, we show that EVAP(4, 3) can be solved in polynomial time, if the input graph is 2-vertex-connected.

2 Definitions

For a subset $V' \subseteq V$ in G, $G - V'$ denotes the subgraph induced by $V - V'$. For an edge set F with $F \cap E = \emptyset$, we denote $G = (V, E \cup F)$ by $G + F$. An edge with end vertices u and v is denoted by (u, v). A *partition* $X_1, \cdots, X_t$ of vertex set V means a family of nonempty disjoint subsets of V whose union is V, and a *subpartition* of V means a partition of a subset of V. For two disjoint subsets of vertices X, $Y \subset V$, we denote by $E_G(X, Y)$ the set of edges, one of whose end vertices is in X and the other is in Y, and also denote $c_G(X, Y) = |E_G(X, Y)|$. A *cut* is defined as a subset X of V with $\emptyset \neq X \neq V$, and the *size* of a cut X is denoted by $c_G(X, V - X)$, which may also be written as $c_G(X)$. A cut with the minimum size is called a *minimum cut*, and its size, denoted by $\lambda(G)$, is called the *edge-connectivity* of G. For a subset X of V, $\{v \in V - X \mid (u, v) \in E$ for some $u \in X\}$ is called the *neighbor set* of X, denoted by $\Gamma_G(X)$. Let $p(G)$ denote the number of components in G. A *separator* of G is defined as a cut S of V such that $p(G - S) > p(G)$ holds and no $S' \subset S$ has this property. If G does not contain K_n, then a separator of the minimum size is called a *minimum separator*, and its size, denoted by $\kappa(G)$, is called the *vertex-connectivity* of G. If G contains the complete graph K_n, we define $\kappa(G) = n - 1$. If $\kappa(G) = 2$, then we call a minimum separator S a *separating pair* in G.

2.1 Edge-Splitting

We introduce an operation of transforming a graph, called *edge-splitting*, which is helpful to solve the edge-connectivity augmentation problem.

Given a multigraph $G = (V, E)$, a designated vertex $s \in V$, vertices $u, v \in \Gamma_G(s)$ (possibly $u = v$) and a nonnegative integer $\delta \leq \min\{c_G(s, u), c_G(s, v)\}$, we construct graph $G' = (V, E')$ from G by deleting δ edges from $E_G(s, u)$ and $E_G(s, v)$, respectively, and adding new δ edges to $E_G(u, v)$: $c_{G'}(s, u) := c_G(s, u) - \delta$, $c_{G'}(s, v) := c_G(s, v) - \delta$, $c_{G'}(u, v) := c_G(u, v) + \delta$, $c_{G'}(x, y) := c_G(x, y)$ for all other pairs $x, y \in V$. We say that G' is obtained from G by *splitting* (s, u) and (s, v) by size δ, and denote the resulting graph G' by $G/(u, v; \delta)$. A sequence of splittings is *complete* if the resulting graph G' does not have any neighbor of s.

The following theorem is proven by Mader [3].

Theorem 2.1 [3] *Let $G = (V, E)$ be a multigraph with a designated vertex $s \in V$ with $c_G(s) \neq 1, 3$ and $\lambda_G(x, y) \geq 2$ for all pairs $x, y \in V$. Then for any edge $(s, u) \in E$ there is an edge $(s, v) \in E$ such that $\lambda_{G/(u,v;1)}(x, y) = \lambda_G(x, y)$ holds for all pairs $x, y \in V - s$.* □

This says that if $c_G(s)$ is even, there always exists a complete splitting at s such that the resulting graph G' satisfies $\lambda_{G'-s}(x, y) = \lambda_G(x, y)$ for every pair of $x, y \in V - s$.

3 EVAP(4, 3) for a 2-Vertex-Connected Graph

We now present a polynomial time algorithm for EVAP(4,3) for a given 2-vertex-connected graph.

Let $\beta(G) \equiv \max\{p(G - S) - 1 + \max[0, \max\{4 - c_G(v_1), 4 - c_G(v_2)\}\} \mid S = \{v_1, v_2\}$ is a separating pair in $G]$. To make a graph G (4,3)-connected, it is necessary to add at least $4 - c_G(X)$ edges to $E_G(X, V - X)$ for each cut X, to add at least $3 - |\Gamma_G(X)|$ edges to $E_G(X, V - X)$ for each cut X with $V - X - \Gamma_G(X) \neq \emptyset$, and to add at least $p(G - S) - 1 + \max[0, \max\{4 - c_G(v_1), 4 - c_G(v_2)\}\}]$ edges to connect components of $G - S$ for each separating pair $S = \{v_1, v_2\}$ in G.

Lower Bound: $\gamma(G) \equiv \max\{\lceil \alpha(G)/2 \rceil, \beta(G)\}$, where

$$\alpha(G) = \max\left\{\sum_{i=1}^{p}(4 - c_G(X_i)) + \sum_{i=p+1}^{q}(3 - |\Gamma_G(X_i)|)\right\}$$

and the max is taken over all subpartitions $\{X_1, \cdots, X_p, X_{p+1}, \cdots, X_q\}$ of V such that $q \geq p \geq 0$ and $V - X_i - \Gamma_G(X_i) \neq \emptyset$, $i = p + 1, \cdots, q$. □

For a subset F of edges in a graph G, we say that two edge $e_1 = (u_1, w_1)$ and $e_2 = (u_2, w_2)$ are *switched* in F if we delete e_1 and e_2 from F, and add edges (u_1, u_2) and (w_1, w_2) to F, and that an edge $e_1 = (u_1, w_1)$ is *shifted* in F, if we delete e_1 from F and add an edge (u_1, w_2) $(w_1 \neq w_2)$ to F. The sketch of our algorithm for solving the EVAP(4, 3) for a 2-vertex-connected graph, denoted by Algorithm EVA3, is given as follows.

Algorithm EVA3

Input: An undirected 2-vertex-connected multigraph $G = (V, E)$.

Output: An undirected multigraph $G^* = G + F$ with $\lambda(G^*) \geq 4$ and $\kappa(G^*) \geq 3$ where the size of new edge set F is the minimum.

Step I. (Adding vertex s and associated edges): After adding a new vertex s, we can add a set F_1 of new edges between s and V so that $|F_1| = \alpha(G)$ and the resulting graph $G_1 = (V \cup \{s\}, E \cup F_1)$ satisfies $c_{G_1}(X) \geq 4$ for all cut $X \subset V$, $|\Gamma_{G_1}(X \cup s)| \geq 3$ for all cut $X \subset V$ with $V - X - \Gamma_{G_1}(X) \neq \emptyset$.

Step II. (Edge-splitting): We find a complete edge-splitting at s in G_1 which preserves the 4-edge-connectivity, according to Theorem 2.1. Ignore the isolated vertex s and denote the resulting graph $G_2 = (V, E \cup F_2)$.

If G_2 is also 3-vertex-connected, then we are done because $|F_2| = |F_1|/2 = \lceil \alpha(G)/2 \rceil$ implies that G_2 is optimally augmented by lower bound $\lceil \alpha(G)/2 \rceil$. Otherwise, go to Step III.

Step III. (Switching and Shifting edges): Now G_2 has separating pairs.

We repeat switching or shifting edges in F_2 so that the resulting graph G_2' satisfies the following properties:
· the 4 edge-connectivity,
· $\kappa_{G_2'}(x, y) \geq 3$ for all $x, y \in V$ with $\kappa_{G_2}(x, y) \geq 3$.
· $p(G_2' - S) < p(G_2 - S)$ for some separating pairs S.

Let $G_3 = (V, E \cup F_3)$ be the resulting graph, where F_3 denotes the final F_2. Then in G_3, there is no separating pair S_1, S_2 such that $S_1 \cap S_2 = \emptyset$.

If G_3 has no separating pair, then we are done, since $|F_3| = \lceil \alpha(G)/2 \rceil$ implies that G_3 is optimally augmented. Otherwise, go to Step IV.

Step IV. (Edge augmentation): Now one of the following (i) or (ii) satisfy:
(i) We can make G_3 3-vertex-connected by adding a set F_4 of $\beta(G) - \lceil \alpha(G)/2 \rceil$ new edges, i.e., we are done since $|F_3| + |F_4| = \beta(G)$ implies that the resulting graph is optimally augmented by lower bound $\beta(G)$.
(ii) The input graph G can be made (4, 3)-connected by adding at most four edges.

Theorem 3.1 *For a 2-vertex-connected multigraph G, G can be made (4, 3)-connected by adding $\gamma(G) = \max\{\lceil \alpha(G)/2 \rceil, \beta(G)\}$ new edges or at most four new edges in polynomial time.* □

References

[1] L. R. Ford and D. R. Fulkerson, Flows in Networks, Princeton University Press, Princeton, N. J., 1962.

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[3] W. Mader, *A reduction method for edge-connectivity in graphs*, Ann. Discrete Math., Vol.3, 1978, pp. 145-164.