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On Mix Efficiency in DEA

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1 Introduction

In the previous paper [1], the author has proposed a slacks-based measure of efficiency (SBM) as defined by

$$\rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{io}}{1 + \frac{1}{s}\sum_{i=1}^{s} s_i^+/y_{io}}. \quad (1)$$

In this paper, two modifications of the SBM model will be developed along with the decomposition of the SBM efficiency into *radial*, *scale* and *mix* efficiencies. This will help us to understand an inefficient DMU in terms of radial (ratio), scale and non-radial (mix) inefficiencies.

2 Input-Oriented SBM Model

We will modify the SBM model by dealing only with the numerator of the expression (1). Then the problem becomes:

$$\begin{aligned}
\text{[SBM-IN]}\ \min\ \rho_{in} &= 1 - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{io} \quad (2)\\
\text{subject to}\ \boldsymbol{x}_o &= X\boldsymbol{\lambda} + \boldsymbol{s}^-\\
\boldsymbol{y}_o &= Y\boldsymbol{\lambda} - \boldsymbol{s}^+\\
\boldsymbol{\lambda} \geq 0,\ \boldsymbol{s}^- &\geq 0,\ \boldsymbol{s}^+ \geq 0.
\end{aligned}$$

This modification emphasizes the input slacks by neglecting the output slacks in the objective function. We call it the *input-oriented* SBM model. Let an optimal solution of [SBM-IN] above be $(\rho_{in}^*, \boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$. Then we have:

Theorem 1 *The optimal objective value ρ_{in}^* of the input-oriented SBM model satisfies*

$$\rho^* \leq \rho_{in}^* \leq \theta^*$$

Definition 1 (SBM-IN efficient)
A DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is SBM-IN efficient, if $\rho_{in}^* = 1$ and $\boldsymbol{s}^{+*} = 0$ for every optimal solution of [SBM-IN].

Regarding SBM-IN-efficiency we have, from the above definition,

Theorem 2 *A DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is SBM-IN efficient, if and only if it is CCR efficient and hence if and only if it is SBM efficient.*

The CCR model evaluates the radial (proportional) measure of efficiency, while the SBM model deals with non-radial sources of inefficiency as represented by slacks $\boldsymbol{s}^-$ and $\boldsymbol{s}^+$. Therefore, it is a matter of concern to reveal the essential difference between them. For this purpose, we focus on the relationship between the input-oriented CCR model (CCR) and the input-oriented SBM model (SBM-IN). We will deal with two cases classified by the optimal values.

[Case 1]
First we investigate the case when the two measures give the same efficiency value for a DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, i.e.,

$$\rho_{in}^* = \theta^*.$$

Let an optimal CCR solution be $(\theta^*, \boldsymbol{\mu}^*, \boldsymbol{t}^{-*}, \boldsymbol{t}^{+*})$. It satisfies

$$\begin{aligned}
\theta^*\boldsymbol{x}_o &= X\boldsymbol{\mu}^* + \boldsymbol{t}^{-*} \quad (3)\\
\boldsymbol{y}_o &= Y\boldsymbol{\mu}^* - \boldsymbol{t}^{+*}\\
\boldsymbol{\mu}^* &\geq 0,\ \boldsymbol{t}^{-*} \geq 0,\ \boldsymbol{t}^{+*} \geq 0.
\end{aligned}$$

From (3) we have

$$\boldsymbol{x}_o = X\boldsymbol{\mu}^* + (1-\theta^*)\boldsymbol{x}_o + \boldsymbol{t}^{-*}.$$

Therefore, $(\boldsymbol{\lambda} = \boldsymbol{\mu}^*, \boldsymbol{s}^- = (1-\theta^*)\boldsymbol{x}_o + \boldsymbol{t}^{-*}, \boldsymbol{s}^+ = \boldsymbol{t}^{+*})$ is feasible for [SBM-IN]. This gives the corresponding objective function value of [SBM-IN] as

$$\begin{aligned}
\rho_{in} &= 1 - \frac{1}{m}\sum_{i=1}^{m}\frac{(1-\theta^*)x_{io} + t_i^{-*}}{x_{io}}\\
&= \theta^* - \frac{1}{m}\sum_{i=1}^{m}\frac{t_i^{-*}}{x_{io}}. \quad (4)
\end{aligned}$$

Since $\rho_{in}^* \leq \rho_{in}$ and $\rho_{in}^* = \theta^*$, the second term on the right-hand side of (4) must be zero. Hence, if $\rho_{in}^* = \theta^*$, it holds,

$$\boldsymbol{t}^{-*} = 0.$$

This means that the optimal CCR-solution has no slacks in inputs. Hence the CCR-projection can be achieved by the proportional reduction $\theta^* \boldsymbol{x}_o$ of inputs, as far as inputs are concerned. Now we will turn to the [SBM-IN] side. The feasible solution $(\boldsymbol{\lambda} = \boldsymbol{\mu}^*, \boldsymbol{s}^- = (1-\theta^*)\boldsymbol{x}_o + \boldsymbol{t}^{-*}, \boldsymbol{s}^+ = \boldsymbol{t}^{+*})$ is also optimal for [SBM-IN] under the assumption $\rho^*_{in} = \theta^*$, since it gives the optimal objective value equal to θ^*. The SBM-IN projection by this optimal solution is achieved, in the inputs part, by

$$\boldsymbol{x}_o - \boldsymbol{s}^- = \boldsymbol{x}_o - (1-\theta^*)\boldsymbol{x}_o = \theta^* \boldsymbol{x}_o.$$

The last term means that this projection is a proportional reduction of inputs. Thus, we have:

Theorem 3 *If, for a DMU* $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, [SBM-IN] *and* [CCR] *have the same optimal objective value, both projections can be achieved by the same proportional reduction of inputs. The converse is also true.*

[Case 2]
We deal with the case $\rho^*_{in} < \theta^*$. According to the preceding theorem, the SBM-IN projection must be carried out by a non-proportional reduction of inputs in this case. Thus, a change in the proportion of input data $\boldsymbol{x}_o$ of the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ must result.

Based on the discussions above, it would be reasonable to define the *input-mix* efficiency as follows,

Definition 2 (input-mix efficiency)
The input-mix efficiency is defined by

$$\theta^*_{mix} = \frac{\rho^*_{in}}{\theta^*}.$$

The input-mix efficiency satisfies $0 \leq \theta^*_{mix} \leq 1$ and equals one if and only if $\rho^*_{in} = \theta^*$ holds.

The CCR efficiency (θ^*) is measured under the constant returns to scale assumption and is called the *technical* efficiency, while the input-oriented BCC efficiency (θ^*_{BCC}) is measured under the variable returns to scale assumption and is called the input-based *pure technical* efficiency. The ratio θ^*/θ^*_{BCC} is named the input-based *scale* efficiency (θ^*_{scale}). Thus, we have a decomposition of the SBM-IN efficiency (ρ^*_{in}) in terms of the pure technical efficiency, the scale efficiency and the input-mix efficiency as

$$\rho^*_{in} = \theta^*_{BCC} \times \theta^*_{scale} \times \theta^*_{mix}. \tag{5}$$

Since the three terms on the right-hand side are bounded by zero and one, we have $\rho^*_{in} = 1$ if and only if $\theta^*_{BCC} = \theta^*_{scale} = \theta^*_{mix} = 1$ holds.

3 Output-Oriented SBM Model

In contrast to the input-oriented model, we can introduce the output-oriented one by dealing with the denominator of the objective function of (1). This results in, for a DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$,

$$\begin{aligned} \text{[SBM-OUT]} \quad \min \ \rho_{out} &= \frac{1}{1 + \frac{1}{s}\sum_{i=1}^{s} s_i^+ / y_{io}} \quad (6) \\ \text{subject to } \boldsymbol{x}_o &= X\boldsymbol{\lambda} + \boldsymbol{s}^- \\ \boldsymbol{y}_o &= Y\boldsymbol{\lambda} - \boldsymbol{s}^+ \\ \boldsymbol{\lambda} &\geq 0, \ \boldsymbol{s}^- \geq 0, \ \boldsymbol{s}^+ \geq 0. \end{aligned}$$

This program can be transformed into a linear program by reversing the objective function and by replacing min by max. Let an optimal solution of [SBM-OUT] be $(\rho^*_{out}, \boldsymbol{\lambda}^*, \boldsymbol{s}^{-*}, \boldsymbol{s}^{+*})$. Then we have:

Theorem 4 *The optimal value ρ^*_{out} of the output-oriented SBM model satisfies*

$$\rho^* \leq \rho^*_{out} \leq \theta^*. \tag{7}$$

We note that even if we assume the variable returns to scale production possibility set by imposing the convexity condition $\sum_{j=1}^{n} \lambda_j = 1$, the above theorem remains valid, after revising ρ^*, ρ^*_{out} and θ^* to cope with the convexity condition. In this case θ^* is the optimal objective value of the output-oriented BCC model, which is not always equal to the input-oriented one.

Similarly to the input-oriented case, we define the output-mix efficiency θ_{mix-O} as the ratio of the output-oriented SBM efficiency ρout^* and the CCR efficiency θ^*:

$$\theta^*_{mix-O} = \frac{\rho^*_{out}}{\theta^*}. \tag{8}$$

Thus, we have a decomposition of the output-oriented SBM efficiency into the pure technical, scale and output-mix efficiencies in the output-orientation as

$$\rho^*_{out} = \theta^*_{BCC-O} \times \theta^*_{scale-O} \times \theta^*_{mix-O}. \tag{9}$$

This decomposition will contribute to finding the sources of the output inefficiency in terms of the (output-based) pure technical, scale and mix inefficiencies.

References

[1] K. Tone, in Abstracts of the Spring National Conference of ORSJ, 1998.