An algorithm for solving the edge-disjoint path problem on tournament graphs

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1 Introduction

Given a connected graph \( G = (V, E) \) and \( K \) pairs of vertices \( (x_i, y_i), \ i = 1, \ldots, K, \) the edge-disjoint path problem asks to construct \( K \) pairwise edge-disjoint paths connecting each pair \( (x_i, y_i) \) from source \( x_i \) to sink \( y_i, \ i = 1, \ldots, K, \) where paths \( P_1, P_2, \ldots, P_i, i \geq 2, \) are edge-disjoint. A tournament graph (tournament for short) is a directed graph such that there is precisely one edge between each pair of vertices. On tournaments, J. Bang-Jensen showed a necessary and sufficient condition for the existence of edge-disjoint \((x_1, y_1), (x_2, y_2)\)-paths and an \( O(n^4) \) time algorithm for examining the existence of such paths where \( n \) is the number of vertices \([1]\). In this paper, we propose an \( O(n^2) \) time algorithm for examining the existence of edge-disjoint \((x_1, y_1), (x_2, y_2)\)-paths and for constructing them, if they exist, using the property of tournaments.

2 Definition

A digraph \( D \) consists of a pair \( V(D), A(D) \) where \( V(D) \) is a finite set of vertices and \( A(D) \) is a set of ordered pairs \( (u, v) \) of vertices, called edges. If an edge \((u, v)\) exists in \( A(D)\), we say that \( u \) dominates \( v \). The number of vertices \( y \in U \subseteq V(D) \) dominated by \( x \) is denoted by \( d^+(x) \). We call \( d^+(x) \) the out-degree of \( x \) and simply is denoted by \( d^+(x) \). Similarly, the number of vertices \( y \in U \subseteq V(D) \) dominating \( x \) is denoted by \( d^-(x) \) and \( d^-(x) \) for short) is called the in-degree of \( x \). A component \( D' \) of a digraph is a maximal subgraph such that for any two vertices \( x, y \) of \( D' \), \( D' \) contains an \((x, y)\)-path and \((y, x)\)-path. A digraph \( D \) is strong if it has only one component.

3 Algorithm

We first describe a property of tournament.

[Property 1] When tournament \( T \) is not strong, it is divided into some components and we can label these components \( T_1, T_2, \ldots, T_i \) such that each vertex of \( T_j \) dominates all vertices of \( T_i \) if \( i < j \).

We say that \( T_1 \) (respectively, \( T_i \)) is the initial component (respectively, the terminal component) of \( T \).

By Property 1, for each degree \( d^+(v) \), if \( v_i \in V(T), v_j \in V(T), i < j \), is satisfied, then \( d^+(v_i) < d^+(v_j) \) holds. Moreover, the following lemma is deduced.

[Lemma 1] If \( d^+(v_i) = d^+(v_j) \) is satisfied, then \( v_i, v_j \) belong to the same component.

J. Bang-Jensen gave the necessary and sufficient condition of the existence of two edge-disjoint \((x_1, y_1), (x_2, y_2)\)-paths in tournament \( T \).

[Definition 1] Let \( T \) be a strong tournament and let \( x_1, y_1, x_2, y_2 \) be four different vertices in \( T \). The 5-tuple \((T, x_1, x_2, y_1, y_2)\) is said to be of Type 1a. There exists a proper subset \( S \subseteq V(T) \) such that \( y_1, y_2 \in S \), \( x_1, x_2 \in S = T - S \) and there is exactly one edge from \( S_2 \) to \( S_1 \) in \( T \).

Type 1b. It is not of Type 1a and there exists a partition \( S_1, S_2, S_3 \) of \( V(T) \) into disjoint non empty subsets with the following conditions. \( y_1 \in S_1, \ y_2 \in S_2, \ y_3 \in S_2, \ y_4 \in S_3 \) for \( i = 1 \) or 2: Vertices in \( S_1 \) dominate all the vertices in \( S_2 \) which again dominate all the vertices in \( S_3 \). There exists exactly one edge from \( S_3 \) to \( S_1 \) and it goes from the terminal component in \( T[S_3] \) to the initial component in \( T[S_1] \).

Type 2r. For some \( r \geq 1 \), there exists a partition \( S_1, S_2, \ldots, S_{2r+1} \) of \( V(T) \) into disjoint non empty subsets with the following conditions. \( y_1 \in S_1, y_2 \in S_2, x_1 \in S_{2r+1}, x_2 \in S_{2r+2} \) for \( i = 1 \) or 2: All the edges between \( S_i \) and \( S_j \) where \( i < j \) go from \( S_i \) to \( S_j \) with the following exceptions: There exists precisely one edge from \( S_{2r+1} \) to \( S_{2r+2} \) and it goes from the terminal component in \( T[S_3] \) to the initial component in \( T[S_2] \).

Type 2r+1. For some \( r \geq 1 \), there exists a partition \( S_1, S_2, \ldots, S_{2r+3} \) of \( V(T) \) into disjoint non empty subsets with the following conditions. \( y_1 \in S_1, y_2 \in S_2, \ x_1 \in S_{2r+2}, x_2 \in S_{2r+3} \) for \( i = 1 \) or 2: All the edges between \( S_i \) and \( S_j \) where \( i < j \) go from \( S_i \) to \( S_j \) with the following exceptions: There exists precisely one edge from \( S_{2r+3} \) to \( S_{2r+2} \) and it goes from the terminal component in \( T[S_3] \) to the initial component in \( T[S_2] \).
Lemma 2 [1] Let $T$ be a tournament and let $x_1, y_1, x_2, y_2$ be different vertices such that $T$ contains an $(x_i, y_i)$-path $i = 1, 2$. Then $T$ has edge-disjoint $(x_1, y_1), (x_2, y_2)$-paths unless $x_1, y_1, x_2, y_2$ all belong to the same component $T_j$ of $T$ and $(T, x_1, x_2, y_1, y_2)$ is of one of the types $1a, 1b, 2r$ or $2r+1$ for some $r \geq 1$, in which case $T$ does not have these paths.

Based on the property and these lemmas, we get the following procedure for examining whether edge-disjoint $(x_1, y_1), (x_2, y_2)$-paths exist or not.

Procedure Check_Existence

begin

(Step 1) Check whether $T$ has an $(x_i, y_i)$-path for $i = 1$ and 2, not necessary edge-disjoint. If not then $T$ does not have edge-disjoint $(x_1, y_1), (x_2, y_2)$-paths and the procedure stops.

(Step 2) $d^+_\text{max}(v) \leftarrow \max\{d^+(w) \mid (v, w) \in E(T)\}$.

(Step 3) Set the degree $d^+_\text{max}(u_i)$ of $u_i$ into array $D^+_i$, $i = 1, \ldots, n$, and sort $D^+_i$ in the order of ascending degree. Calculate the value of $T[i]$, $S[i]$ and $\text{Diff}[i]$.

(Step 4) Check whether $x_1, y_1, x_2$ and $y_2$ all belong to the same component $T_j$ of $T$. If not then $T$ has edge-disjoint $(x_1, y_1), (x_2, y_2)$-paths and the procedure stops.

(Step 5) Let $T = T_j$ (namely, throw away the rest of $T$).

(Step 6) Assume that $d^+(x_i) \leq d^+(x_{3-i})$.

In the following steps, we examine whether $T$ is divided into some component or not by exchanging the direction of an edge $(v, w)$. In the order of ascending degree, check Condition 1 below and get a vertex $v$ satisfying the condition first. If there is no vertex satisfying Condition 1, edge-disjoint $(x_1, y_1), (x_2, y_2)$-paths exist and stops.

Condition 1: at least one of $\text{Diff}[1], \ldots, \text{Diff}[\text{max}(d^+_\text{max}(v))]$ has 1 and its index is not less than $i_{\text{min}}(d^+(x_i))$.

On $\text{Diff}[\text{max}(v)], \ldots, \text{Diff}[\text{max}(d^+_\text{max}(v))] - 1]$, find index $i$ such that $\text{Diff}[i] = 1$. Assume here that an $(v, w)$ is selected and $\text{Diff}[i] = \text{Diff}[j] = \ldots = \text{Diff}[k] = 1$, $i < j < \ldots < k$ hold. By exchanging the direction of the edge $(v, w)$, $T$ is not strong and an induced subgraph $D\{v_1, v_2, \ldots, v_n\}$ is a component $T_1$, $D\{v_{i+1}, \ldots, v_j\} = T_2$, ..., $D\{v_{i+k}, \ldots, v_n\} = T_k$.

(Step 7) Find a component including $x_i$. We here assume that $x_i \in V(T_k)$.

(Case I) When $x_3 - i$ also exists in $T_k$.

(I.I) If either $y_i$ or $y_{3-i}$ belongs to $T_1, \ldots, T_k$ edge-disjoint $(x_1, y_1), (x_2, y_2)$-paths exist in $T$ and the procedure stops.

(I.II) If both $y_i$ and $y_{3-i}$ exist in $T_{k+1}, \ldots, T_i$, edge-disjoint $(x_1, y_1), (x_2, y_2)$-paths do not exist in $T$ and the procedure stops.

(Case II) When $x_3 - i$ exists in $T_j, k < j$.

(II.I) If $y_{3-i}$ exists in $V(T_j)$, edge-disjoint $(x_1, y_1), (x_2, y_2)$-paths exist in $T$ and the procedure stops.

(II.II) If $y_{3-i}$ exists in $V(T_j)$, $j' < j$, edge-disjoint $(x_1, y_1), (x_2, y_2)$-paths exist in $T$ and the procedure stops.

(II.III) If $y_{3-i}$ exists in $V(T_j)$, $j' > j$, edge-disjoint $(x_1, y_1), (x_2, y_2)$-paths do not exist in $T$ and the procedure stops.

(Case III) When $x_3 - i, y_i$ and $y_{3-i}$ all belong to $T_i$.

Let $T = T_i$ and $x_i = w$, namely, remove vertices and edges which do not belong to $T_i$ from $T$. Actually, it is sufficient for the procedure to changes the value of arrays $D^+, \text{Diff}, d^+_\text{max}, V^+$.

Go to Step 6.

end. □

Lemma 3 On tournament, Procedure Check_Existence can examine whether edge-disjoint $(x_1, y_1), (x_2, y_2)$-paths exist or not. □

Theorem 1 Procedure Check_Existence can examine the existence of edge-disjoint $(x_1, y_1), (x_2, y_2)$-paths in $O(n^2)$ time. □

We obtain the following result though we do not write details because of the lack of space.

Theorem 2 Procedure Find_Path can find edge-disjoint $(x_1, y_1), (x_2, y_2)$-paths in $O(n^2)$ time. □

References