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On Optimal Stopping of a Sequence of Random Variables with Fuzziness

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ファジィ数に値をとる確率変数は、ファジィ確率変数と呼ばれている。本発表では、ファジィ確率変数列の最適停止問題を考え、ファジィ確率変数のランダム性は通常の期待値でファジィ性は可能性測度での評価を行う。ある条件の下に、ファジィ確率変数列の最適停止問題に対する最適停止時刻を得られる。またその条件を仮定できない場合にも、ε-最適停止時刻を与えることができる。

1. Notations

Let $(\Omega, \mathcal{M}, P)$ be a probability space. Let $\mathbf{R}$ be the set of all real numbers and let $\mathbf{N}$ be the set of all nonnegative integers. A fuzzy set $\tilde{a}$ is called a fuzzy number if the membership function $\tilde{a} : \mathbf{R} \mapsto [0,1]$ is normal, upper-semicontinuous, convex and has a compact support. $\mathcal{R}$ denotes the set of all fuzzy numbers. We write the α-cut ($\alpha \in [0,1]$) of a fuzzy number $\tilde{a} \in \mathcal{R}$ by $\tilde{a}_\alpha := [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$. A map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a fuzzy random variable if the maps $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are measurable for all $\alpha \in [0,1]$, where $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] := \{x \in \mathbf{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$ is the α-cut of the fuzzy number $\tilde{X}(\omega)$ for $\omega \in \Omega$. For an integrably bounded fuzzy random variable $\tilde{X}$, the expectation $E(\tilde{X})$ is a fuzzy number defined by

$$E(\tilde{X})(x) := \sup_{\alpha \in [0,1]} \min \left\{ \alpha, 1_{E(\tilde{X})_\alpha}(x) \right\} \quad \text{for } x \in \mathbf{R},$$

where the closed intervals $E(\tilde{X})_\alpha$ are

$$E(\tilde{X})_\alpha := \left[\int_\Omega \tilde{X}_\alpha^-(\omega)\,\mathrm{d}P(\omega), \int_\Omega \tilde{X}_\alpha^+(\omega)\,\mathrm{d}P(\omega) \right], \quad \alpha \in [0,1].$$

Further, for a sub-σ-field $\mathcal{N}(\subset \mathcal{M})$, the conditional expectation $E(\tilde{X}|\mathcal{N})$ is a fuzzy random variable defined by

$$E(\tilde{X}|\mathcal{N})(\omega)(x) := \sup_{\alpha \in [0,1]} \min \left\{ \alpha, 1_{E(\tilde{X}_\alpha|\mathcal{N})(\omega)}(x) \right\} \quad \text{for } \omega \in \Omega,\ x \in \mathbf{R},$$

where $E(\tilde{X}_\alpha|\mathcal{N})(\omega) := [E(\tilde{X}_\alpha^-|\mathcal{N})(\omega), E(\tilde{X}_\alpha^+|\mathcal{N})(\omega)]$, and $E(\tilde{X}_\alpha^\pm|\mathcal{N})$ are the conditional expectations such that

$$\int_\Lambda E(\tilde{X}_\alpha^\pm|\mathcal{N})(\omega)\,\mathrm{d}P(\omega) = \int_\Lambda \tilde{X}_\alpha^\pm(\omega)\,\mathrm{d}P(\omega) \quad \text{for all } \Lambda \in \mathcal{N}.$$

2. An optimal stopping problem

We deal with an optimal stopping problem for a sequence of fuzzy random variables. Let $\{\tilde{X}_n\}_{n\in\mathbf{N}}$ be a sequence of integrably bounded fuzzy random variables such that $E\left(\sup_n \tilde{X}_{n,0}^+\right) < \infty$, where $\tilde{X}_{n,0}^+(\omega)$ is the right-end of the 0-cut of the fuzzy number $\tilde{X}_n(\omega)$ for $n \in \mathbf{N}$. For $n \in \mathbf{N}$, $\mathcal{M}_n$ denotes the smallest σ-field on Ω generated by all random variables $\tilde{X}_{k,\alpha}^\pm$ ($k = 0,1,2,\cdots,n; \alpha \in [0,1]$). A map $\tau : \Omega \mapsto \mathbf{N} \cup \{\infty\}$ is called a stopping time if $\{\tau = n\} \in \mathcal{M}_n$ for all $n \in \mathbf{N}$. For a finite stopping time τ, we define a fuzzy random variable $\tilde{X}_\tau$ by $\tilde{X}_\tau(\omega) := \tilde{X}_n(\omega)$ if $\tau(\omega) = n$ for $n \in \mathbf{N}$.

A fuzzy goal is a fuzzy set $\varphi : \mathbf{R} \mapsto [0,1]$ which is a continuous and nondecreasing function with $\lim_{x\to-\infty}\varphi(x) = 0$ and $\lim_{x\to\infty}\varphi(x) = 1$. For a stopping time τ, we define a fuzzy expectation of the fuzzy numbers $E(\tilde{X}_\tau)$ by

$$\tilde{E}(E(\tilde{X}_\tau)) := \fint_{\mathbf{R}} E(\tilde{X}_\tau)(x)\ \mathrm{d}\tilde{P}(x) = \sup_{x\in\mathbf{R}} \min\{E(\tilde{X}_\tau)(x), \varphi(x)\},$$

where $\tilde{P}$ is the possibility measure generated by the density φ and $\fint \mathrm{d}\tilde{P}$ denotes Sugeno integral. The fuzzy expectation implies the degree of satisfaction of fuzzy rewards $E(\tilde{X}_\tau)$. Then the fuzzy goal $\varphi(x)$ means a kind of utility function for expected payoffs x. We define an optimal fuzzy reward $\tilde{V}$ as follows: We consider the optimal fuzzy reward

$$\bigvee_{\tau:\ \text{stopping times}} E(\tilde{X}_\tau),$$

where $\vee$ means the supremum induced from the fuzzy max order as follows: For fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{R}$, we define the maximum $\tilde{a}\vee\tilde{b}$ with respect to the fuzzy max order by the fuzzy number whose α-cuts are

$$(\tilde{a}\vee\tilde{b})_\alpha = [\max\{\tilde{a}_\alpha^-, \tilde{b}_\alpha^-\}, \max\{\tilde{a}_\alpha^+, \tilde{b}_\alpha^+\}], \quad \alpha \in [0,1].$$

We define

$$V_\alpha^- := \sup_{\tau:\ \text{stopping times}} E(\tilde{X}_\tau)_\alpha^- \quad \text{and} \quad V_\alpha^+ := \sup_{\tau:\ \text{stopping times}} E(\tilde{X}_\tau)_\alpha^+$$

for $\alpha \in [0,1]$, and then we can define a fuzzy number $\tilde{V} \in \mathcal{R}$ such that

$$\tilde{V}_\alpha^- := \begin{cases} \lim_{\alpha'\uparrow\alpha} V_{\alpha'}^- & \text{for } \alpha > 0 \\ \lim_{\alpha'\downarrow 0} V_{\alpha'}^- & \text{for } \alpha = 0 \end{cases} \quad \text{and} \quad \tilde{V}_\alpha^+ := \begin{cases} \lim_{\alpha'\uparrow\alpha} V_{\alpha'}^+ & \text{for } \alpha > 0 \\ \lim_{\alpha'\downarrow 0} V_{\alpha'}^+ & \text{for } \alpha = 0. \end{cases}$$

Problem 1. Find a stopping time τ^* such that

$$\tilde{E}(E(\tilde{X}_{\tau^*})) = \tilde{E}(\tilde{V}).$$

Then, τ^* is called an optimal stopping time, and a real number x^* is called an optimal expected payoff if it attains the supremum of the fuzzy expectation, i.e.

$$\tilde{E}(\tilde{V}) = \sup_{x\in\mathbf{R}} \min\{\tilde{V}(x), \varphi(x)\} = \min\{\tilde{V}(x^*), \varphi(x^*)\}.$$

Let $n \in \mathbf{N}$. Define

$$Z_{n,\alpha}^- := \operatorname*{ess\,sup}_{\tau:\ \text{stopping times},\, \tau\geq n} E(\tilde{X}_{\tau,\alpha}^- | \mathcal{M}_n) \quad \text{and} \quad Z_{n,\alpha}^+ := \operatorname*{ess\,sup}_{\tau:\ \text{stopping times},\, \tau\geq n} E(\tilde{X}_{\tau,\alpha}^+ | \mathcal{M}_n)$$

for $\alpha \in [0,1]$, where $\tilde{X}_{\tau,\alpha}(\omega) = [\tilde{X}_{\tau,\alpha}^-(\omega), \tilde{X}_{\tau,\alpha}^+(\omega)]$ is the α-cut of the fuzzy number $\tilde{X}_\tau(\omega)$. Put a grade α^* by

$$\alpha^* := \sup\{\alpha \in [0,1] \mid \varphi_\alpha^- \leq \tilde{V}_\alpha^+\},$$

where $\varphi_\alpha = [\varphi_\alpha^-, \infty)$ for $\alpha \in (0,1)$, and $\sup\emptyset = 0$.

Assumption A. $V_{\alpha^*}^+ = \tilde{V}_{\alpha^*}^+$.

Theorem 1. *Suppose that Assumption A holds. Define a stopping time*

$$\sigma^*(\omega) := \inf\{n \mid Z_{n,\alpha^*}^+(\omega) = \tilde{X}_{n,\alpha^*}^+(\omega)\}, \quad \omega \in \Omega,$$

where $\inf\emptyset = +\infty$. *If* σ^* *is finite, then* σ^* *is an optimal stopping time for Problem 1. The optimal expected payoff is* $x^* = \varphi_{\alpha^*}^-$.