Determination of Optimal Repair-Cost Limit Replacement Strategy by Lorenz Transform Method

T. Dohi† (01307065), N. Kaio‡ (01105445) and S. Osaki† (01002265)
†Hiroshima University, ‡Hiroshima Shudo University

1. INTRODUCTION

In this paper, we consider a repair-cost limit replacement problem and develop the graphical solution method to determine the optimal repair-cost limit which minimizes the expected cost per unit time in the steady-state, using the Lorenz transform of the repair-cost distribution function. The model under consideration is somewhat different from the existing ones [1, 2]. Also, the method proposed can be applied to an estimation problem of the optimal repair-cost limit from empirical cost data.

2. MODEL DESCRIPTION

Consider a single-unit repairable system, where each spare is provided only by an order after a lead time $L(>0)$ and each failed unit is repairable. The original unit begins operating at time 0 and the mean time to failure for each unit is $m_f(>0)$. When the unit has failed, the decision maker wishes to determine whether he or she should repair it or order a new spare. If the decision maker estimates that the repair is completed within a prespecified cost limit $v_0 \in [0,\infty)$, then the repair is started immediately at the failure time. The mean repair time is $m_s(>0)$ when the repair cost does not exceed v_0. On the other hand, if the decision maker estimates that the repair cost exceeds the cost v_0, then the failed unit is scrapped and a new spare unit is ordered. Then the spare unit is delivered after the lead time L. Without any loss of generality, it is assumed that the unit once repaired is presumed as good as new and that the time required for replacement is negligible.

The repair cost for each unit is unknown and the decision maker has a *subjective* probability distribution function $H(v)$ on the repair cost, with density $h(v)$ and finite mean $m_m(>0)$. Suppose that the distribution function $H(v)$ is arbitrary, continuous and strictly increasing in $v \in [0,\infty)$, and has an inverse function, *i.e.* $H^{-1}(\cdot)$. Under these model assumptions, define the interval from the start of the operation to the following start as one cycle. The costs considered in this paper are the following;

k_f (>0): a cost per unit shortage time.

c (>0): a cost for each order.

We make the following additional assumptions:

(A-1) $m_s > L$.

(A-2) $k_f m_s < k_f L + c$.

The assumptions (A-1) implies that the mean repair time m_s is strictly longer than the lead time. In the assumption (A-2), the shortage cost when the repair cost does not exceed v_0 is less than the total cost when the new spare is ordered.

Let us formulate the expected cost during one cycle. If the decision maker judges that a new spare unit should be ordered, then the ordering cost for one cycle is $c\overline{H}(v_0)$, where $\overline{H}(\cdot) = 1-H(\cdot)$. In this case, the expected shortage cost is $k_f L\overline{H}(v_0)$. On the other hand, if he or she selects the repair option, the expected repair cost is $\int_0^{v_0} v dH(v)$ and the expected shortage cost is $k_f m_s H(v_0)$. Thus the total expected cost for one cycle is

$$E_C(v_0) = \int_0^{v_0} v dH(v) + k_f\{m_s H(v_0) + L\overline{H}(v_0)\} + c\overline{H}(v_0). \quad (1)$$

Also, the mean time of one cycle is

$$E_T(v_0) = m_f + m_s H(v_0) + L\overline{H}(v_0). \quad (2)$$

It may be appropriate to adopt an expected cost per unit time in the steady-state over an infinite planning horizon. The total expected cost per unit time in the steady-state is, from the renewal reward argument,

$$TC(v_0) = \lim_{t\to\infty} \frac{\mathrm{E}[\text{the total cost on } (0,t]]}{t} = E_C(v_0)/E_T(v_0). \quad (3)$$

Then the problem is to determine the optimal repair-cost limit $v_0{}^*$ such as

$$TC(v_0^*) = \min_{0\le v_0<\infty} TC(v_0). \quad (4)$$

3. GRAPHICAL METHOD

In stead of differentiating $TC(v_0)$ with respect to v_0 directly, we here employ the following graphical method. Define the Lorenz transform of the repair-cost distribution $p \equiv H(v)$ by

$$\phi(p) = \frac{1}{m_m} \int_0^p H^{-1}(v)dv, \quad (0 \le p \le 1). \tag{5}$$

Then the curve $\mathcal{L} = (p, \phi(p)) \in [0,1] \times [0,1]$ is called the *Lorenz curve*. From the simple algebraic manipulation, we have

THEOREM 3.1: Suppose that the assumption (A-1) holds. The minimization problem in Eq.(4) is equivalent to

$$\min_{0 \le p \le 1} : M(p, \phi(p)) \equiv \frac{\phi(p) + \xi}{p + \eta}, \tag{6}$$

where

$$\xi = \frac{cm_s - \{k_f(m_s - L) - c\}m_f}{m_m(m_s - L)}, \tag{7}$$

$$\eta = \frac{m_f + L}{m_s - L}. \tag{8}$$

Consequently, the optimal repair-cost limit is determined by $p^* = H(v_0^*)$ which minimizes the tangent slope from the point $\mathrm{B} = (-\eta, -\xi) \in (-\infty, 0) \times (-\infty, 0)$ to the curve $\mathcal{L}$ in the plane $(x, y) \in (-\infty, +\infty) \times (-\infty, +\infty)$ under (A-2).

More precisely, we prove the uniqueness of the optimal repair-cost limit.

THEOREM 3.2: (1) Suppose that the assumptions (A-1) and (A-2) hold. Then there exists a unique optimal solution $p^* = H(v_0^*)$ $(0 < v_0^* < \infty)$ minimizing $M(p, \phi(p))$, where p^* is given by the x-coordinate in the point of contact for the curve $\mathcal{L}$ from the point B.

4. COMPETING REPAIR-PERSONS PROBLEM

Suppose that there are two repair-persons with different repair abilities. We classify two repair-persons into Repair-person 1 and Repair-person 2, respectively. Their repair costs X_1 and X_2 are non-negative random variables with distribution functions $H_j(v)$ $(j = 1, 2)$ and the same finite mean $1/m_m$, respectively. We require the following definition on the stochastic ordering.

DEFINITION 4.1: (1) X_1 is usually stochastic-ordered with respect to X_2 (denoted as $X_1 \le_{st} X_2$) if $\overline{H_1}(v) \le \overline{H_2}(v)$.
(2) X_1 is star-shaped stochastic-ordered with respect to X_2 (denoted as $X_1 \le_* X_2$) if $H_2^{-1}(H_1(v))/v$ is increasing in $v \in (0, H_1^{-1}(1))$.

The following theorem can be proved applying the result by Chandra and Singpurwalla [3].

THEOREM 4.2: Define the optimal repair-cost limits for two repair-persons with the same mean repair time $1/m_m$ as $v_{01}^* = H_1^{-1}(p_1^*)$ and $v_{02}^* = H_2^{-1}(p_2^*)$, respectively, where p_1^* and p_2^* are the solutions for Eq. (6) with $H_j(v)$ $(j = 1, 2)$. If the repair cost for Repair-person 1 is smaller than that for Repair-person 2 in the usual stochastic ordering, then $v_{01}^* \le v_{02}^*$.

5. STATISTICAL ESTIMATION METHOD

Based on the graphical idea in Section 3, we propose a statistical method to estimate the optimal repair-cost limit replacement policy. Suppose that the optimal repair-cost limit has to be estimated from an ordered complete sample $0 = x_0 \le x_1 \le x_2 \le \cdots \le x_n$ of repair cost data from an absolutely continuous repair-cost distribution H, which is unknown. The estimator of $H(v) = p$ is the empirical distribution given by

$$H_n(x) = \begin{cases} i/n & \text{for} \quad x_i \le x < x_{i+1}, \\ 1 & \text{for} \quad x_n \le x, \end{cases} \tag{9}$$

where $i = 0, 1, 2, \cdots, n-1$. Then the *empirical Lorenz curve* is defined as

$$\phi_i \equiv \sum_{i=1}^{[np]} x_i / \sum_{i=1}^{n} x_i, \tag{10}$$

where $[a]$ is the greatest integer in a. Plotting the point $(i/n, \phi_i)$, $(i = 0, 1, 2, \cdots, n)$, and connecting them by line segments, we obtain the empirical Lorenz curve $\mathcal{L}_n \in [0,1] \times [0,1]$.

As empirical counterpart of THEOREM 3.1, we propose a non-parametric estimator of the optimal repair-cost limit.

THEOREM 5.1: The optimal repair-cost limit can be estimated by $\hat{v}_n = x_{i^*}$, where

$$\left\{ i^* \mid \min_{0 \le i \le n} \frac{\phi_i + \xi}{i/n + \eta} \right\}. \tag{11}$$

REFERENCES

[1] T. Dohi, H. Koshimae, N. Kaio and S. Osaki, "Geometrical Interpretations of Repair Cost Limit Replacement Policies", *International Journal of Reliability, Quality and Safety Engineering*, vol. 4, pp. 309-333 (1997).

[2] T. Dohi, N. Kaio and S. Osaki, "A Graphical Method to Repair-Cost Limit Replacement Policies with Imperfect Repair", *Mathematical and Computer Modelling* (in press).

[3] Chandra, M. and Singpurwalla, N. D., "Relationship between some notions which are common to reliability and economics", *Mathematics of Operations Research*, vol. 6, pp. 113-121 (1981).