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A logical interpretation for the eigenvalue method in AHP : Why is a weight vector in AHP calculated by the eigenvalue method ?

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1 Introduction

Saaty [2] proposed to determine the relative weights of items and/or alternatives (hereafter call items only) in AHP by the eigenvalue method. The eigenvalue method is widely used (e.g., [4]). However, no evident justification has been given for applying the eigenvalue method in a pairwise comparison matrix $A = (a_{ij})$. This paper presents a logical justification for the eigenvalue method in AHP by means of optimization/equilibrium models.

2 Self-evaluation and non-self-evaluation

For every element a_{ij} of a pairwise comaprison matrix A, we define the meaning of a_{ij} by (the value of the i^{th} item)/(the value of the j^{th} item).

We assume that the values of all items are represented by positive real numbers. Then, it follows that every ratio a_{ij} is positive and $a_{ii} = 1$ $(i, j = 1, \ldots, n)$. From the definition of a_{ij}, $a_{ij} = 1/a_{ji}$ $(1 \leq i \leq n,\ 1 \leq j \leq n)$. Note that the validity of Theorem 2 below is independent with the property of $a_{ij} = 1/a_{ji}$.

In our framework of AHP, every item is evaluated by itself, and assigned a positive real number (call self-evaluation value, w_i).

Proposition 1 *$a_{ij}w_j$ represents the evaluation value of the i^{th} item from the viewpoint of j^{th} item when the self-evaluation value of the j^{th} item is given by w_j.*

By averaging $a_{ij}w_j$ over $j \neq i$, we get: $\sum_{j=1, j\neq i}^{n} a_{ij}w_j/(n-1)$. We define it the non-self-evaluation value of the i^{th} item.

We interpret a pairwise comparison matrix A and an element a_{ij} as a conversion table and a conversion ratio from j to i, respectively. The non-self-evaluation value of the i^{th} item is the average of $n-1$ non-self-evaluation values which are converted into the evaluation value of the i^{th} item by others' self-evaluation values according to the conversion table A.

3 Some equilibrium models for a pairwise comparison matrix

We can develop several indices of a discrepancy between the self-evaluation value and the corresponding non-self-evaluation value for each i. For an index, the set of the discrepancies of all items are denoted $\{\gamma_1, \ldots, \gamma_n\}$. The distribution of these discrepancies $\gamma_1, \ldots, \gamma_n$ is then evaluated by several criteria (e.g., minimum, maximum and variance).

Here we use the ratio of self and non-self-evaluation values as the discrepancy index, that is, $\gamma_i = \dfrac{(\text{the } i^{th} \text{ non-self-evaluation value})}{(\text{the } i^{th} \text{ self-evaluation value})}$ for $i = 1, \ldots, n$, which we call i^{th} overestimation rate. Note that γ_i depends on w i.e., $\gamma_i(w)$. We introduce two evaluation criteria of the $\gamma_i(w)$s' distribution:

$$\begin{aligned} f_1(w) &\equiv \max\{\gamma_1(w), \ldots, \gamma_n(w)\} \\ f_2(w) &\equiv \min\{\gamma_1(w), \ldots, \gamma_n(w)\} \end{aligned}$$

From $f_1(w)$ and $f_2(w)$, we define

$$f_3(w) \equiv f_1(w) - f_2(w)$$

as a criterion of variation among $\gamma_1, \ldots, \gamma_n$. Three following optimization models may improve differences among n overestimation rates $\gamma_1, \ldots, \gamma_n$:

$$\min_{w>0} f_1(w) = \min_{w>0}\max \left\{ \frac{\sum_{j=1, j\neq 1}^{n} a_{1j}w_j}{(n-1)w_1}, \ldots, \frac{\sum_{j=1, j\neq n}^{n} a_{nj}w_j}{(n-1)w_n} \right\} \quad (1)$$

$$\max_{w>0} f_2(w) = \max_{w>0}\min \left\{ \frac{\sum_{j=1, j\neq 1}^{n} a_{1j}w_j}{(n-1)w_1}, \ldots, \frac{\sum_{j=1, j\neq n}^{n} a_{nj}w_j}{(n-1)w_n} \right\} \quad (2)$$

and

$$\min_{w>0} f_3(w) = \min_{w>0} f_1(w) - f_2(w) \qquad (3)$$

The models (1),(2), and (3) are based on the following idea. By increasing/decreasing a self-evaluation value w_i, its corresponding overestimation rate $\gamma_i(w)$ is decreased/increased, and all other overestimation rates are increased/decreased. We can interpret the model (1) as the following aggregation process : A set (vector) w of self-evaluation values provides a difference among n overestimation rates $\{\gamma_1, \ldots, \gamma_n\}$. Therefore, in the criterion of the model (1), each item with the largest overestimation rate is intended to increase its self-evaluation value; simultaneously all other items are intended to decrease its self-evaluation value. By repeating this aggregation process, we reach an equilibrium overestimation rate $\hat{\lambda}$ (the derivation, see below) such that $\hat{\lambda} = \gamma_1 = \gamma_2 = \cdots = \gamma_n$. In the same manner, we can get the equilibria for the models (2) and (3). Therefore these three optimization models can be called equilibrium models.

4 Optimal solution

To show the equivalence between the eigenvector with the principal eigenvalue of A and an optimal solution for any equilibrium model, the following famous theorem can be used:

Theorem 2 (Frobenius's Theorem) *Let $\lambda_{\max}$ be positive and the maximum absolute value of eigenvalues for an $n \times n$ matrix A whose element is nonnegative. For every n-vector w whose element is positive,*

$$\min\left\{\frac{\sum_{j=1}^n a_{1j}w_j}{w_1}, \ldots, \frac{\sum_{j=1}^n a_{nj}w_j}{w_n}\right\} \le \lambda_{\max} \le \max\left\{\frac{\sum_{j=1}^n a_{1j}w_j}{w_1}, \ldots, \frac{\sum_{j=1}^n a_{nj}w_j}{w_n}\right\}.$$

Furthermore, if a matrix A is irreducible,

$$\max_{w>0}\min\left\{\frac{\sum_{j=1}^n a_{1j}w_j}{w_1}, \ldots, \frac{\sum_{j=1}^n a_{nj}w_j}{w_n}\right\} = \lambda_{\max} = \min_{w>0}\max\left\{\frac{\sum_{j=1}^n a_{1j}w_j}{w_1}, \ldots, \frac{\sum_{j=1}^n a_{nj}w_j}{w_n}\right\}.$$

For the n-identify matrix I, let $\hat{A}$ and $\hat{a}_i$ be $A - I$ and the i^{th} row vector of $\hat{A}$, respectively. Then every element of $\hat{A}$ is nonnegative and irreducible, and the three equilibrium models (1), (2) and (3) are rewritten as $\min_{w>0}\max\{\hat{a}_1 w/w_1, \ldots, \hat{a}_n w/w_n\}$, $\max_{w>0}\min\{\hat{a}_1 w/w_1, \ldots, \hat{a}_n w/w_n\}$ and $\min_{w>0}\{\max\{\hat{a}_1 w/w_1, \ldots, \hat{a}_n w/w_n\} - \min\{\hat{a}_1 w/w_1, \ldots, \hat{a}_n w/w_n\}\}$, respectively. Let $\hat{\lambda}_{\max}$ and $\hat{w}$ be the principal eigenvalue of $\hat{A}$ and the corresponding eigenvector. Since $\hat{\lambda}_{\max}$ is the simple root of the characteristic equation of $\hat{A}$, we get the following two theorems:

Theorem 3 *Every model (1),(2) and (3) has the common optimal solution $\hat{w}$. The optimal values of (1) and (2) are $\hat{\lambda}_{\max}$ and that of (3) is 0.*

Theorem 4 *Let $\lambda_{\max}$ and w be the principal eigenvalue of A and the corresponding eigenvector, respectively. Then*

$$\frac{\lambda_{\max} - 1}{n - 1} = \hat{\lambda}_{\max} \quad \textit{and} \quad w = \hat{w}. \qquad (4)$$

From Theorem 4 it follows that the consistency index $C.I. = \hat{\lambda}_{\max} - 1$.

5 Concluding remarks

The current method can be interpreted as follows. The meaning of the eigenvalue method is to obtain the equilibrium solution for the difference among all discrepancies between a self-evaluation value and its corresponding non-self-valuation value.

We can extend the current models (1), (2) and (3) to the new models for AHP with incomplete pairwise comparisons and get a different weight vector from those of Harker method [1] and TS method [3]. Thus, the current approach provides some generalizations of these weighing methods used in AHP.

References

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