1999年度日本オペレーションズ・リサーチ学会
秋季研究発表会

Maximization of the Ratio of Two Convex Quadratic Functions over a Polytope

02103210 東京工業大学 *後藤 順哉 GOTOH Jun-ya
01102370 東京工業大学 今野 浩 KONNO Hiroshi

1 Introduction

Lo&MacKinlay[4] formulate a portfolio selection problem in the form of a quadratic fractional problem with a few linear constraints. It is not easy to calculate a global optimal solution because it is not a concave-convex-type fractional programming problem. In this paper, we generalize their formulation and develop an algorithm for a nonconvex quadratic fractional problem, which includes Lo&MacKinlay's problem as a special case. The problem can be formulated as follows :

$$(\mathrm{P})\quad \begin{array}{|ll} \text{maximize} & f(\boldsymbol{x}) \equiv \dfrac{\boldsymbol{x}'Q\boldsymbol{x}}{\boldsymbol{x}'P\boldsymbol{x}} \\ \text{subject to} & \boldsymbol{x} \in X \subset R^n, \end{array}$$

where P, Q : $n \times n$ positive semi-definite matrices, X : a polytope.

We assume $\boldsymbol{x}'P\boldsymbol{x} > 0, \forall \boldsymbol{x} \in X$. Under these assumptions, a well-known framework for nonlinear fractional programming problems addressed by Dinkelbach[1] is applicable to (P).

2 Basic Analysis

Let us start with the problem (P) when $X = R^n$. The following proposition provides a geometric image of the objective function $f(\boldsymbol{x})$.

PROPOSITION 1 (Gantmacher [2]) *: The maximum of $f(\boldsymbol{x})$ with respect to $\boldsymbol{x} \in R^n$ is given by the largest eigenvalue λ^* of the matrix $B \equiv P^{-1}Q$, and is attained by the eigenvector $\boldsymbol{x}^*$ associated with the largest eigenvalue of B.*

Therefore, if the problem (P) has no constraint, it suffices to seek the maximal eigenvalue and corresponding eigenvector. However, a problem with constraints requires global optimization techniques.

Let us introduce a function, $F(\boldsymbol{x}, \lambda)$, $\boldsymbol{x} \in X$, $\lambda \in R$ as follows :

$$F(\boldsymbol{x}, \lambda) \equiv \boldsymbol{x}'(Q - \lambda P)\boldsymbol{x}.$$

PROPOSITION 2 (Dinkelbach [1]) *If there exists $\lambda^* > 0$ which satisfies the equation :*

$$F(\boldsymbol{x}^*, \lambda^*) \equiv \max_{\boldsymbol{x} \in X} F(\boldsymbol{x}, \lambda^*) = 0,$$

then $\boldsymbol{x}^*$ is an optimal solution of (P).

In other words, the problem (P) is equivalent to the problem of finding λ satisfying the above condition.

PROPOSITION 3 *Let P and Q be $n \times n$ positive definite matrices and let us denote $\lambda_{max} = \max f(\boldsymbol{x})$, $\lambda_{min} = \min f(\boldsymbol{x})$, respectively. Then the following statements hold :*

$\forall \lambda > \lambda_{max}$, $(Q - \lambda P)$ *is negative definite,*
$\forall \lambda < \lambda_{min}$, $(Q - \lambda P)$ *is positive definite,*
$\forall \lambda \in (\lambda_{min}, \lambda_{max})$, $(Q - \lambda P)$ *is indefinite.*

For simplicity, we denote

$$\pi(\lambda) \equiv \max_{\boldsymbol{x} \in X} F(\boldsymbol{x}, \lambda).$$

From Proposition 3, once λ is given, we must solve a nonconvex optimization problem, viz., $\pi(\lambda)$. In this paper, we apply a decomposition branch and bound method, *rectangular subdivision algorithm*, whose usefulness is proved by Phong *et al.*[5]. A symmetric matrix $Q - \lambda P$ can be transformed into the separable (d.c.) form by a standard decomposition technique, e.g.,

$$\boldsymbol{x}'(Q - \lambda P)\boldsymbol{x} \longrightarrow \sum_{i=1}^{l} c_i y_i^2 - \sum_{j=1}^{n-l} d_j z_j^2,$$

where $c_i > 0$, $i = 1, \ldots, l$, $-d_j < 0$, $j = 1, \ldots, n-l$ are eigenvalues of $Q - \lambda P$.

Let $\Omega \in (\boldsymbol{y}, \boldsymbol{z})$ be the polytope associated with X. We can easily obtain a rectangle S containing Ω :

$$S = \{(\boldsymbol{y}, \boldsymbol{z}) | L_i \le y_i \le U_i,\ i = 1, \ldots, l\}.$$

An overestimating function on S can be defined as

$$g(\boldsymbol{y}, \boldsymbol{z}) = \sum_{i=1}^{l} \phi_i(y_i) - \sum_{j=1}^{n-l} d_j z_j^2,$$

where

$$\phi_i(y_i) = c_i(U_i + L_i)y_i - c_i U_i L_i,\ i = 1, \ldots, l$$

The maximization of $g(\boldsymbol{y}, \boldsymbol{z})$ over $\Omega \cap S$ which is a concave quadratic program gives an upperbound of the optimal value of F on $\Omega \cap S$ for fixed λ.

Let us remark that l is smaller when λ is larger. Clearly, $\pi(\lambda)$ is a decreasing convex function of $\lambda \in R$. Therefore, binary search or Newton's method can be applied to search λ such that $\pi(\lambda) = 0$. However, we apply the following more efficient procedure introduced by Ibaraki[3].

Interpolated BINARY (Ibaraki[3])

Step 1 Find λ' and λ'' such that $\pi(\lambda') > 0$ and $\pi(\lambda'') < 0$, and solve $\pi(\lambda')$ and $\pi(\lambda'')$. Let $\lambda^1 = \lambda'$ and $\lambda^2 = \lambda''$.

Step 2 Compute $\hat{\pi}(\lambda)$ defined below and its root $\bar{\lambda}$ satisfying $\hat{\pi}(\bar{\lambda}) = 0$. Solve $\pi(\bar{\lambda})$. If $|\pi(\bar{\lambda})| \le \varepsilon$, halt. Otherwise go to Step 3.

Step 3 If $\pi(\bar{\lambda}) > 0$, let $\lambda^1 \equiv \bar{\lambda}$ and return to Step 2. Otherwise let $\lambda^u \equiv \bar{\lambda}$ and return to Step 2.

$$\begin{aligned}
\hat{\pi}(\lambda) &= \begin{cases} \boldsymbol{x}^{u\prime} P \boldsymbol{x}^u (\lambda^u - \lambda) + a(\lambda^u - \lambda)^b + \pi(\lambda^u), \\ \qquad\qquad \text{if } \boldsymbol{x}^{u\prime} P \boldsymbol{x}^u + \Delta\pi \neq 0, \\ \boldsymbol{x}^{u\prime} P \boldsymbol{x}^u (\lambda^u - \lambda) + \pi(\lambda^u), \\ \qquad\qquad \text{otherwise}, \end{cases} \\
a &= -(\boldsymbol{x}^{u\prime} P \boldsymbol{x}^u + \Delta\pi)/(\lambda^u - \lambda^l)^{b-1}, \\
b &= (\boldsymbol{x}^{u\prime} P \boldsymbol{x}^u - \boldsymbol{x}^{l\prime} P \boldsymbol{x}^l)/(\boldsymbol{x}^{u\prime} P \boldsymbol{x}^u + \Delta\pi), \\
\Delta\pi &= (\pi(\lambda^u) - \pi(\lambda^l))/(\lambda^u - \lambda^l),
\end{aligned}$$

where $\boldsymbol{x}^u$ the optimal solution of $\pi(\lambda^u)$.

3 An Algorithm for Solving (P)

The algorithm to be used in this paper is the following :

step 0 Let $\varepsilon > 0$ be some tolerance. Set $k = 1$.

step 1 Compute the largest eigenvalue λ_{max} and the corresponding eigenvector $\boldsymbol{x}(\lambda_{max})$ of $P^{-1}Q$. If $\boldsymbol{x}(\lambda_{max}) \in X$, terminate : $\boldsymbol{x}(\lambda_{max})$ is the optimal solution. Otherwise, goto step 2.

step 2 Solve the maximization problem $\pi(\lambda_{max})$ by an ordinary algorithm for convex program. If $\pi(\lambda_{max}) + \varepsilon > 0$ then terminate : the solution is ε-optimal, else select a λ_1 by the above procedure and goto step 3.

step 3 Decompose $Q - \lambda_k P$ into a diagonal matrix, which has eigenvalues of $Q - \lambda_k P$ as diagonal elements and transform $F(\boldsymbol{x}, \lambda_k)$ into the equivalent separable form such as $\sum c_i y_i^2 - \sum d_j z_j^2$, where $c_i, d_j > 0$, and transform also X into another polytope associating with $(\boldsymbol{y}, \boldsymbol{z}) \in R^n$. Solve it *by rectangular subdivision algorithm*[5]. If $|\pi(\lambda_k)| < \varepsilon$ then terminate with optimal solution. Otherwise, select λ_{k+1} by the Interpolated BINARY procedure and set $\lambda_k \leftarrow \lambda_{k+1}$ and goto step 3.

4 Example : Maximizing Predictability Portfolio

As an example, let us outline the formulation of the Maximizing Predictability Portfolio Problem, which is formulated by Lo&MacKinlay[4]. Assume that there are n assets and let $\boldsymbol{x}$ be the investment weight for n assets. Let $R_t = (R_{t1}, \ldots, R_{tn})'$, $t = 1, \ldots, T$ denote the vectors of historical return, and let $\bar{R}_t = (\bar{R}_{t1}, \ldots, \bar{R}_{tn})'$, $t = 1, \ldots, T$ denote the vector of return data forecasted by a linear regression model.

Then, the predictability of the portfolio can be defined as follows :

$$\frac{Var\,[\boldsymbol{x}'(\bar{R}_t - E[R_t])]}{Var\,[\boldsymbol{x}'(R_t - E[R_t])]} \equiv \frac{\boldsymbol{x}'Q\boldsymbol{x}}{\boldsymbol{x}'P\boldsymbol{x}}.$$

where $Var[\cdot]$: variance operator.

Here, P, Q stand for variance-covariance matrices of observed return and forecasted return, respectively. In this case, the objective value is at most 1. Therefore, we know $\lambda_{max} \le 1$.

As usual, portfolio has to satisfy some constraints. For example, we can define it as :

$$\{\, \boldsymbol{x} \mid \sum_{j=1}^{n} x_j = 1,\ \sum_{j=1}^{n} r_j x_j = \rho,\ \boldsymbol{0} \le \boldsymbol{x} \le \boldsymbol{u},\ \sum_{j \in J_k} x_j \le C_k, k = 1, \ldots, K \,\}.$$

where

ρ	:	subjective expected return,
r_j	:	expected return of asset j,
$\boldsymbol{u}$	:	upper bound for the weight of investment,
J_k	:	index set for some asset class,
C_k	:	some constant less than 1.

In many cases, constraints for portfolio selection problem can be represented as a polytope, where the above algorithm is applicable. On the other hand, the scale of the problem varies case by case. Under practical environment, we need to solve $n = 5 \sim 30$.

We will show computational results at the time of presentation.

References

[1] Dinkelbach, W.(1967), "On Nonlinear Fractional Programming", *Management Science*, **13**(7), 492-498.

[2] Gantmacher, F.R.(1959), "The Theory of Matrices", Chelsea Pub. Co.

[3] Ibaraki, T.(1983), "Parametric Approaches to Fractional Programs", *Mathematical Programming*, **26**, 345-362.

[4] Lo, A. and MacKinlay, C.(1997), "Maximizing Predictability in the Stock and Bond Markets", *Macroeconomic Dynamics*, **1**, 102-134.

[5] Phong, T.Q., An, L.T.H., Tao, P.D.(1995), "Decomposition branch and bound method for globally solving linearly constrained indefinite quadratic minimization problems", *Operations Research Letters*, **17**, 215-220.