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Minimal Cost Rebalancing under Concave Transaction Costs and Minimal Transaction Unit Constraints

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1. Introduction

This paper is concerned with minimal cost rebalancing problem under concave transaction costs and minimal transaction unit (MTU) constraints.

Recently, we proposed a branch and bound algorithm for solving a concave cost portfolio optimization problem under the mean-absolute deviation framework [3]. We used a piecewise linear underestimating function for the concave cost function and solved the resulting linear subproblems by a branch and bound method using a well-designed problem reduction technique. We showed in [3] that this algorithm generates a good solution in a very efficient manner.

Here, we formulate the rebalancing problem as the piecewise-concave cost minimization problem and apply the same branch and bound algorithm using a piecewise linear convex underestimation strategy for calculating a optimal solution. We will extend this algorithm to even more difficult class of problem where the amount of transaction is constrained to be the integer multiple of minimal transaction units. The problem thus becomes a concave minimization problem with integer constraints on the variables.

2. Minimal Cost Rebalancing under Concave Transaction Costs

Let $\boldsymbol{x}^0$ be the portfolio at hand and assume that an investor wants to rebalance the portfolio in such a way that the new portfolio $\boldsymbol{x}$ satisfies the condition that its expected rate of return $E[R(\boldsymbol{x})]$ is greater than some constant ρ and that the risk is $W[R(\boldsymbol{x})]$ is smaller than some constant w.

As in the mean absolute deviation (MAD) model [2,3], we will employ the absolute deviation as the measure of risk in this rebalancing problem.

$$\begin{aligned} W[\boldsymbol{x}] &= E[\,|R(\boldsymbol{x}) - E[R(\boldsymbol{x})]\,|\,] \\ &= \sum_{t=1}^{T} p_t \Big| \sum_{j=1}^{n} (r_{jt} - r_j) x_j \Big|. \end{aligned}$$

The expected return is given by

$$E[R(\boldsymbol{x})] = r(\boldsymbol{x}) = \sum_{j=1}^{n} r_j x_j.$$

Let S be an investable set. Then the minimum cost rebalancing problem can be formulated as follows.

$$\begin{array}{|lll} \text{minimize} & c(\boldsymbol{x}) & \\ \text{subject to} & E[R(\boldsymbol{x})] \;\geq\; \rho, & \\ & W[R(\boldsymbol{x})] \;\leq\; w, & \\ & \boldsymbol{x} \in S, & \end{array}$$

where $c(\boldsymbol{x})$ is a concave transaction cost associated with the purchase/sale of stock S and

$S = \{\boldsymbol{x} \,|\, 0 \leq x_j \leq \alpha_j,\ j = 1, \cdots, n\}.$

Let us introduce a new set of variables,

$\boldsymbol{v} = \boldsymbol{x} - \boldsymbol{x}^{o}.$

Then the above problem can be represented as follows:

$$(P)\left|\begin{array}{ll} \text{minimize} & \sum_{j=1}^{n} c_j(v_j) \\ \text{subject to} & (\boldsymbol{v}, \boldsymbol{z}) \in G, \\ & -x_j^0 \leq v_j \leq \alpha_j - x_j^0, \end{array}\right.$$

where

$$\begin{aligned} G = \ &\{ (\boldsymbol{v}, \boldsymbol{z}) \mid \sum_{j=1}^{n} r_j v_j \geq \rho - \sum_{j=1}^{n} r_j x_j^0, \\ & z_t = p_t \sum_{j=1}^{n} [(r_{jt} - r_j) v_j + (r_{jt} - r_j) x_j^0], \\ & \sum_{t=1}^{T} |z_t| \leq w,\ t = 1, \cdots, T,\ \boldsymbol{v} \in V \}, \end{aligned}$$

where V is the set of feasible $\boldsymbol{v}$'s corresponding to S, and $c_j(v_j)$ is the cost associated with purchasing v_j units (if $v_j > 0$) and selling v_j units (if $v_j < 0$) of j th asset. Let us assume again that $c_j(v_j)$ is piecewise concave and that $c_j(0) = 0$ for all j.

We construct a branch and bound algorithm which will be explained in Section 3. Let (H_k) be a subproblem :

$$(P_k)\left|\begin{array}{ll} \text{minimize} & \sum_{j=1}^{n} c_j(v_j) \\ \text{subject to} & (\boldsymbol{v}, \boldsymbol{z}) \in G, \\ & \beta_j^k \leq v_j \leq \alpha_j^k. \end{array}\right.$$

We will approximate the function $c_j(v_j)$ in the interval $[\alpha_j^k, \beta_j^k]$ by a piecewise-linear convex underestimating function $c_j^k(v_j)$ and define a relaxed subproblem :

$$(Q_k)\left|\begin{array}{ll} \text{minimize} & \sum_{j=1}^{n} c_j^k(v_j) \\ \text{subject to} & (\boldsymbol{v}, \boldsymbol{z}) \in G, \\ & \beta_j^k \leq v_j \leq \alpha_j^k. \end{array}\right.$$

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which can be reduced to a linear programming problem by using a standard method.

3. A Branch and Bound Algorithm

1° $\boldsymbol{P} = \{(P_0)\}$, $\hat{f} = +\infty$, $k = 0$.

2° If $\boldsymbol{P} = \{\phi\}$, then goto 9°; Otherwise goto 3°.

3° Choose a problem $(P_k) \in \boldsymbol{P}$:

$$(P_k) \left| \begin{array}{ll} \text{minimize} & f(\boldsymbol{v}) = \sum_{j=1}^{n} c_j(v_j) \\ \text{subject to} & (\boldsymbol{v}, \boldsymbol{z}) \in G, \quad \boldsymbol{\beta}^k \leq \boldsymbol{v} \leq \boldsymbol{\alpha}^k. \end{array} \right.$$

$$\boldsymbol{P} = \boldsymbol{P} \backslash \{(P_k)\}.$$

4° Let $c_j^k(v_j)$ be a linear underestimating function of $c_j(v_j)$ over the interval $\beta_j^k \leq v_j \leq \alpha_j^k$, $(j = 1, \cdots, n)$ and define a linear programming problem

$$(Q_k) \left| \begin{array}{ll} \text{minimize} & g_k(\boldsymbol{v}) = \sum_{j=1}^{n} c_j^k(v_j) \\ \text{subject to} & (\boldsymbol{v}, \boldsymbol{z}) \in G, \quad \boldsymbol{\beta}^k \leq \boldsymbol{v} \leq \boldsymbol{\alpha}^k. \end{array} \right.$$

If (Q_k) is infeasible then go to 2°. Otherwise let $\boldsymbol{v}^k$ be an optimal solution of (Q_k).
If $|f(\boldsymbol{v}^k) - g_k(\boldsymbol{v}^k)| > \varepsilon$ then goto 8°. Otherwise let $f_k = f(\boldsymbol{v}^k)$.

5° If $f_k > \hat{f}$ then goto 7°; Otherwise goto 6°.

6° If $\hat{f} = f_k$; $\hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}^k$ and eliminate all the subproblems (P_t) for which $g_t(\boldsymbol{v}^t) \geq \hat{f}$.

7° If $g_k(\boldsymbol{v}^k) \geq \hat{f}$ then goto 2°. Otherwise goto 8°.

8° Let
$c_s(v_s^k) - c_s^k(v_s^k) = \max\{c_j(v_j^k) - c_j^k(v_j^k)\}$,

$$\begin{aligned} S_{l+1} &= S_k \cap \{\boldsymbol{v} \mid \beta_s^k \leq v_s \leq (\beta_s^k + \alpha_s^k)/2\}, \\ S_{l+2} &= S_k \cap \{\boldsymbol{v} \mid (\beta_s^k + \alpha_s^k)/2 \leq v_s \leq \alpha_s^k\}, \end{aligned}$$

and define two subproblems :

(P_{l+1}) maximize $\{f(\boldsymbol{v}) | \ (\boldsymbol{v}, \boldsymbol{z}) \in G, \ \boldsymbol{v} \in S_{l+1}\}$.
(P_{l+2}) maximize $\{f(\boldsymbol{v}) | \ (\boldsymbol{v}, \boldsymbol{z}) \in G, \ \boldsymbol{v} \in S_{l+2}\}$.

$\boldsymbol{P} = \boldsymbol{P} \cup \{P_{l+1}, P_{l+2}\}, k = k + 1$ and goto 3°.

9° Stop : $\hat{\boldsymbol{v}}$ is an ε optimal solution of (P_0).

Theorem 3.1 : $\hat{\boldsymbol{v}}$ converges to an ε - optimal solution of (P_0) as $k \to \infty$.

Proof : See e.g. [1,4] □
To accelerate convergence, we may replace the bisection scheme by the ω - subdivision scheme [1], in which the interval $[\beta_s^k, \alpha_s^k]$ is divided into two subintervals $[\beta_s^k, v_s^k]$ and $[v_s^k, \alpha_s^k]$ where v_s^k is the s th component of the optimal solution $\boldsymbol{v}^k$ of (Q_k). This subdivision scheme usually accelerate convergence, though it is not theoretically guaranteed.

4. Computational Experiments

We conducted numerical tests of the algorithm proposed in this paper using monthly data of 200 stocks chosen from NIKKEI 225 Index. We used the breadth first rule for choosing subproblems in Step 3^0 of the branch and bound Algorithm and ω– subdivision strategy throughout the test. Also we choose $\varepsilon = 10^{-5}$ in our computation. We choose appropriate level of ρ and w in view of the new efficient frontier calculated by a new set of 36 monthly data.

As expected, the performance of the algorithm is more or less the same as the that of portfolio construction problem reported in [3]. This rebalancing problem proves that an investor sells/buys a smaller number of assets when the terminal risk is the same as the original portfolio and the elapse of time is short enough. However, it increases as the elapse of time and the discrepancy of ρ and w from the original portfolio increases.
(Computational results will be presented at the time of presentation)

5. Conclusions

We showed in this paper that the portfolio rebalancing problem under concave transaction costs can be solved in a practical amount of time. The success depends upon the use of mean absolute deviation model, elaboration of the classical branch and bound method using ω - subdivision strategy.

References

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