1999年度日本オペレーションズ・リサーチ学会
秋季研究発表会

An Extension of the Two Phase Process in the CCR Model

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1. Introduction

The two phase process usually adopted in DEA aims at obtaining the maximum sum of slacks (input excesses and output shortfalls). Hence it may occur that the projection of an inefficient DMU onto the efficient frontier is very far from the observed values in the sense that the input mix and output mix of the projected point are very different from those of the observations. It may be desirable to achieve the opposite approach: to project onto an efficient point which has input mix and output mix as near as possible to the observations. We propose an extension of the two phase process to achieve this.

2. Phase III Process

First, we apply the two phase process to the data set and find the set R composed of CCR-efficient DMUs, i.e., DMUs with $\theta^*=1$, $s^{-*}=0$, $s^{+*}=0$ for every optimal solution. We call R the "peer set."

Then we go into Phase III process as follows. For each inefficient DMU$_o$, solve the following LP:

[Phase III]

$$\min\ w \tag{1}$$

subject to

$$\theta^* x_{io} = \sum_{j\in R} x_{ij}\lambda_j + \delta_i^- x_{io} \quad (i=1,\ldots,m) \tag{2}$$

$$y_{ro} = \sum_{j\in R} y_{rj}\lambda_j - \delta_r^+ y_{ro} \quad (r=1,\ldots,s) \tag{3}$$

$$\delta_i^- \le p \quad (i=1,\ldots,m) \tag{4}$$

$$\delta_r^+ \le q \quad (r=1,\ldots,s) \tag{5}$$

$$p \le w,\quad q \le w \tag{6}$$

$$\lambda_j \ge 0,\ \delta_i^- \ge 0,\ \delta_r^+ \ge 0,\quad (\forall j,i,r)$$

where θ^* is the optimal objective value obtained in Phase I.

The above LP can be interpreted as follows. The left side of (2) is the radially reduced value of input i of DMU$_o$, which is expressed as the sum of the weighted sum of inputs in the peer set and the slacks as designated by $\delta_i^- x_{io}$ on the right. Hence δ_i^- denotes the relative deviation ratio from the observed x_{io}. The inequality (4) bounds the maximum of δ_i^- $(i=1,\ldots,m)$ by p. In the same way, we bound the maximum deviation ratio of output by q in (5). Finally, we put the bound w to p and q, and minimize w in the objective function (1). Therefore, we want to find slacks with the minimum deviation ratio from the observed values. (This is a kind of min-max problem.)

Let an optimal solution be $(\boldsymbol{\lambda}^*, s^{-*}, s^{+*}, p^*, q^*, w^*)$. Then the following projection puts DMU$_o$ on the efficient frontier.

$$\widehat{x}_{io} \leftarrow (\theta^* - \delta_i^{-*})x_{io}\ (i=1,\ldots,m)$$
$$\widehat{y}_{ro} \leftarrow (1+\delta_r^{+*})y_{ro}\ (r=1,\ldots,s).$$

We notice that this projection is "units invariant" in the sense that δ_i^{-*} and δ_r^{+*} exhibit no effects from a change of units for input i and output r, as can be seen from (2) and (3). A similar procedure is proposed in Coelli (1998).

3. An Example

We will apply the Phase III process to the following problem in Table 1 with 5 DMUs (A, B, C, D and E), 3 inputs (x_1, x_2 and x_3) and a single unitized output (y). First we applied the CCR model

Table 1: Problem for Phase III Process

DMU	x_1	x_2	x_3	y
A	1	2	1	1
B	1	1	2	1
C	2	10	5	1
D	2	5	10	1
E	2	10	10	1

with Phase I and Phase II processes and obtained results displayed in Table 2. DMUs A and B are CCR-efficient and DMU A is the only reference to inefficient DMUs C, D and E. By observing the input mix in Table 1, C is similar to A rather than B so the result that C's reference is A is reasonable. However, D's input mix is similar to B rather than A and E's is likely to be similar to the combination of A and B. After these observations we applied the Phase III process as follows, using the peer set R={ A, B} :

DMU C: The LP for C is as follows,

$$\min w$$
subject to
$$0.5 \times 2 = 1 = \lambda_1 + \lambda_2 + 2\delta_1^-$$
$$0.5 \times 10 = 5 = 2\lambda_1 + \lambda_2 + 10\delta_2^-$$
$$0.5 \times 5 = 2.5 = \lambda_1 + 2\lambda_2 + 5\delta_3^-$$
$$1 = \lambda_1 + \lambda_2 - \delta^+$$
$$\delta_1^- \leq p,\ \delta_2^- \leq p,\ \delta_3^- \leq p,\ \delta^+ \leq q$$
$$p \leq w,\ q \leq w,$$

where all variables are constrained to be non-negative.

An optimal solution is given by

$$w^* = p^* = 0.3,\ \lambda_1^* = 1,\ \delta_2^{-*} = 0.3,\ \delta_3^{-*} = 0.3,$$

with all other variables zero. This solution is the same as the CCR solution, the maximum deviation ratio being 30% as in Table 2. The Phase III process projects C onto A.

DMU D: The LP for D is as follows,

$$\min w$$
subject to
$$0.5 \times 2 = 1 = \lambda_1 + \lambda_2 + 2\delta_1^-$$
$$0.5 \times 5 = 2.5 = 2\lambda_1 + \lambda_2 + 5\delta_2^-$$
$$0.5 \times 10 = 5 = \lambda_1 + 2\lambda_2 + 10\delta_3^-$$
$$1 = \lambda_1 + \lambda_2 - \delta^+$$
$$\delta_1^- \leq p,\ \delta_2^- \leq p,\ \delta_3^- \leq p,\ \delta^+ \leq q$$
$$p \leq w,\ q \leq w,$$

where all variables are constrained to be non-negative.

An optimal solution is given by

$$w^* = p^* = 0.3,\ \lambda_2^* = 1,\ \delta_2^{-*} = 0.3,\ \delta_3^{-*} = 0.3,$$

with all other variables zero. This solution is different from the CCR solution. The Phase III process projects D onto B. The maximum deviation ratio 30% is less than that of the CCR model 40% which is given by the ratio of excess $s_3^{-*}/x_{3D} = 4/10 = 0.4(40\%)$.

DMU E: The LP for E is as follows,

$$\min w$$
subject to
$$0.5 \times 2 = 1 = \lambda_1 + \lambda_2 + 2\delta_1^-$$
$$0.5 \times 10 = 5 = 2\lambda_1 + \lambda_2 + 10\delta_2^-$$
$$0.5 \times 10 = 5 = \lambda_1 + 2\lambda_2 + 10\delta_3^-$$
$$1 = \lambda_1 + \lambda_2 - \delta^+$$
$$\delta_1^- \leq p,\ \delta_2^- \leq p,\ \delta_3^- \leq p,\ \delta^+ \leq q$$
$$p \leq w,\ q \leq w,$$

where all variables are constrained to be non-negative.

An optimal solution is given by

$$w^* = p^* = 0.35,\ \lambda_1^* = 0.5,\ \lambda_2^* = 0.5,$$
$$\delta_2^{-*} = 0.35,\ \delta_3^{-*} = 0.35,$$

with all other variables zero. This solution is different from the CCR solution. The Phase III process projects D onto the mid-point of A and B. The maximum deviation ratio 35% is less than that of the CCR model 40% which is given by the ratio of excess $s_3^{-*}/x_{3E} = 4/10 = 0.4(40\%)$.

The above results suggest that the Phase III process is suitable for finding a set of similar reference DMUs among the efficient DMUs, while keeping the CCR score at the level of the Phase I result.

Reference

T. Coelli (1998) "A Multi-stage Methodology for the Solution of Oriented DEA Models," *Operations Research Letters*, 22, pp.143-149.

Table 2: CCR-Score, Reference set, Slacks and % Change

DMU	Score	Ref.	s_1	s_2	s_3	s_1/x_1	s_2/x_2	s_3/x_3
A	1	A	0	0	0	0%	0%	0%
B	1	A	0	0	0	0%	0%	0%
C	0.5	A	0	3	1.5	0%	30%	30%
D	0.5	A	0	0.5	4	0%	10%	40%
E	0.5	A	0	3	4	0%	30%	40%