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A Spectral Analysis for a $MAP/D/N$ Queue

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1 Introduction

In an ATM network, information is partitioned into fixed-size packets. To deal with a system of a versatile input process of packets, we consider a $MAP/D/N$ queueing system.

Let $D(z)$ be the $M \times M$ matrix function of a MAP. Let $A(z)$ be the matrix generating function of arrivals during the constant service time. When we formulate a $MAP/D/N$ queue as an $M/G/1$ type Markov chain, there are MN boundary states. Then the transition probability matrix is represented as blocked $MN \times MN$ matrixes and a large memory and huge computation time are required when N is large. Three types of spectral analysis for a $MAP/D/N$ queueing system are considered. The first type is to obtain MN zero points of $\det(z^N I_M - A(z))$ in the unit disk and corresponding right null vectors of $(z^N I_M - A(z))$. The second is to obtain eigenvalues and eigenvectors of $D = D(1)$. From the first and second types of spectral analysis, the stationary probability of boundary states and the mean queue length at service completion epochs in equilibrium are obtained. The third is to obtain eigenvalues and eigenvectors of $D(z)$ at $|z| = 1$. From the third type of spectral analysis we consider the (discrete) Fourier Transform (FT) method and obtain the stationary probability vector.

2 Zero points

$$p(z)(z^N I_M - A(z)) = \sum_{s=0}^{N-1} p_s(z^N B_s(z) - z^s A(z)).$$

Assumption 1. *In the unit disk all zero points* $z_0, \cdots, z_{MN-1}$ $(z_0 = 0)$ *of the equation*

$$\det(z^N I_M - e^{hD(z)}) = 0 \tag{1}$$

are distinct.

Assumption 2. *All eigenvalues* $\omega_{0,0}, \cdots, \omega_{0,M-1}(\omega_{0,0} = 0)$ *of the infinitesimal generator* D *are distinct.*

Proposition 1 *The necessary and sufficient condition that* z *is a zero point of* $\det(z^N I_M - e^{hD(z)})$ *in the open unit disk* ($|z| < 1$) *is that there exists a pair* (α, s) *such that* $s \in \{0, \cdots, N-1\}$ *and* $\alpha \in \mathcal{A} \equiv \{\alpha : |\alpha + \delta| < \delta\}$ *is a zero point of the equation*

$$\det(\alpha I_M - hD(e^{(\alpha + \mathrm{i}2\pi s)/N})) = 0, \tag{2}$$

where i *is the imaginary unit and* $\delta = \max_j(-hd_{jj}(0))$. *In this case* $z = e^{(\alpha+\mathrm{i}2\pi s)/N}$.

Moreover, for $s = 0$, *(2) has one zero point* $\alpha = 0$ *and* $M - 1$ *zero points in* $\mathcal{A}$ *and for* $s = 1, \cdots, N-1$ *(2) has* M *zero points in* $\mathcal{A}$.

From Proposition 1, the problem solving MN zero points $\det(z^N I_M - e^{hD(z)})$ in (1) is divided into N problems to solve M zero points $\det(\alpha I_M - hD(e^{(\alpha+i2\pi s)/N}))$. The computational time of solving zero points is proportional to N.

Proposition 2 *Suppose that z_i and $\boldsymbol{w}_i$ ($\boldsymbol{w}_0 = \boldsymbol{e}_M$) are a zero point of $\det(z^N I_M - e^{hD(z)})$ in the unit disk and the corresponding right null vector of $z^N I_M - e^{hD(z)}$, respectively.*
(i) z_i is an eigenvalue of $C(\boldsymbol{g})$ and $\bar{\boldsymbol{w}}_i$ is the corresponding eigenvector. That is,

$$C(\boldsymbol{g}) = \bar{W}\Delta(\boldsymbol{z})\bar{W}^{-1}.$$

(ii) z_i^N is an eigenvalue of G and $\bar{\boldsymbol{w}}_i$ is the corresponding eigenvector. That is,

$$G = \bar{W}\Delta(\boldsymbol{z}^{(N)})\bar{W}^{-1}.$$

Proposition 3 *The vector $\bar{\boldsymbol{p}}_0 = (\boldsymbol{p}_0, \cdots, \boldsymbol{p}_{N-1})$ satisfies*

$$\sum_{s=0}^{N-1} \boldsymbol{p}_s(B_s(z_i) - z_i^s I_M)\boldsymbol{w}_i = 0 \qquad (3)$$

$$i = 1, \cdots, MN-1.$$

The solution space of homogeneous linear equations (3) is one-dimensional.

Theorem 1 *The vector $\bar{\boldsymbol{p}}_0$ is uniquely determined by the system of homogeneous linear equations*

$$\sum_{s=0}^{N-1} \boldsymbol{p}_s(B_s(z_i) - z_i^s I_M)\boldsymbol{w}_i = 0$$

$$i = 1, \cdots, MN-1,$$

and the nonhomogeneous equation

$$\sum_{s=0}^{N-1} \boldsymbol{p}_s \sum_{l=0}^{N-s-1} F^l(-D_0^{-1})\boldsymbol{e}_M = \frac{1}{\xi_{0,0}}.$$

The mean queue length L at departure epochs is given by the spectral method.

3 Stationary distribution

The inversion formula is

$$\boldsymbol{p}_n \approx \frac{1}{K}\sum_{k=0}^{K-1} \boldsymbol{p}(e^{i2\pi k/K})e^{-i2\pi kn/K}.$$

Assumption 3. *For any fixed $\hat{z}$ ($|\hat{z}| = 1$), eigenvalues $\theta_0, \cdots, \theta_{M-1}$ of $D(\hat{z})$ are distinct.*

Now fix any $\hat{z}$ in the unit circle. Let $\boldsymbol{\phi}_\nu$ ($\nu = 0, \cdots, M-1$) be the left eigenvector of $D(\hat{z})$ corresponding to an eigenvalue θ_ν. And as the $M \times M$ matrix we put $\Phi = (\boldsymbol{\phi}_0^{\boldsymbol{T}}, \cdots, \boldsymbol{\phi}_{M-1}^{\boldsymbol{T}})^{\boldsymbol{T}}$. Since $Y(\hat{z}) = U(\hat{z})^{-1}\Delta(\boldsymbol{\xi}(\hat{z}))U(\hat{z})$, an eigenvalue of $Y(\hat{z})$ corresponding an eigenvector $\boldsymbol{\phi}_\nu$ is

$$\eta_\nu = \theta_\nu e^{h\theta_\nu}(\hat{z}^N - e^{h\theta_\nu})^{-1}.$$

Put $\boldsymbol{\eta} = (\eta_0, \cdots, \eta_{M-1})$. Then we have

$$\boldsymbol{p}(\hat{z}) = \sum_{s=0}^{N-1} \boldsymbol{p}_s \sum_{l=0}^{N-s-1} \hat{z}^{s+l}F^l(-D_0^{-1})\Phi^{-1}\Delta(\boldsymbol{\eta})\Phi.$$

4 Waiting time

The distribution function $P_W(x)$ of the stationary waiting time is given by

$$\begin{aligned} P_W(kh-t) &= \lim_{n\to\infty} P\{W_n \le kh-t\} \\ &= \sum_{l=0}^{kN-1} \boldsymbol{r}_{kN-l-1}P_l(t)\boldsymbol{e}_M, \quad (4) \end{aligned}$$

where $\boldsymbol{r}_\nu = \sum_{l=0}^{\nu} \boldsymbol{q}_l$. Substituting $x = kh-t$ into (4), we have

$$P_W(x) = \sum_{l=0}^{kN-1} \boldsymbol{r}_{kN-l-1}P_l(kh-x)\boldsymbol{e}_M.$$