A Diffusion Approximation for a Single Server System with Finite Number of Sources

01013080 NTT Information Sharing Platform Laboratories ISHIZUKA Mika
01304590 NTT Service Integration Laboratories TAKAHASHI Yoshitaka

1 Introduction

In recently-developed TCP/IP networks, modeling the buffering (queueing) situation leads to a state-dependent queueing network with finite number of jobs (population).

An exact approach to the state-dependent queueing network becomes very hard because of the non-product form structure[1]. We then take a diffusion approximation approach, which is much simpler than the exact approach.

The main goal of this paper is to propose a new diffusion approximation for the GI/GI/1//N system, and to examine the accuracy of our method by some numerical examples.

2 GI/GI/1//N model

We consider a GI/GI/1//N system, where N represents the number of sources. The thinking time in each source is independently and generally distributed with a mean of $\frac{1}{\lambda}$ and a coefficient variance of K_a. The service time is independently and generally distributed with a mean of $\frac{1}{\mu}$ and a coefficient variance of K_s. Customers are served individually, based on the first-in-first-out (FIFO) discipline. Hereafter, the server and the waiting room with an infinite capacity are together called the system.

3 Diffusion approximation

In this section, we propose a diffusion approximation for the GI/GI/1//N system. We approximate the process $Q(t)$(the number of customers in the system) by a homogeneous diffusion process $X(t)$ with elementary return (ER) boundary.

Denote by $f(x,t)$ the p.d.f of $X(t)$, i.e., $f(x,t)dx \triangleq P\{x < Q(t) \leq x+dx\}$ and introduce $f(x)$ by $f(x) \triangleq \lim_{t\to\infty} f(x,t)$.

Then, $f(x)$ satisfies the following equations[2]:

$$\frac{1}{2}\frac{d^2}{dx^2}a(x)f(x) - \frac{d}{dx}b(x)f(x) = -N\lambda p_0 \delta(x-1), \quad (1)$$

$$\frac{1}{2}\frac{d}{dx}a(x)f(x) - \frac{d}{dx}b(x)f(x)|_{x=0} = N\lambda p_0, \quad (2)$$

$$\lim_{x\downarrow 0} f(x) = 0, \quad (3)$$

where $a(x)$ and $b(x)$ are the diffusion parameters, $\delta(x)$ signifies the Dirac density function concentrated at $x=0$, and p_0 denotes the probability mass at the origin.

Piecewise constant (PC)[2] and piecewise linear (PL)[3] have been proposed to determine the diffusion parameters for state-dependent queueing systems.

In PC, the diffusion parameters are defined as follows:

$$\begin{aligned} b(x) &\triangleq b_k = \lambda U(k) - \mu, \ k-1 < x \leq k, \\ a(x) &\triangleq a_k = \lambda K_a^2 U(k) + \mu K_s^2, \\ &\quad k-1 < x \leq k, k = 1, \cdots, N, \end{aligned}$$

where $U(x) = N-k, k-1 < x \leq k, k = 1, \cdots, N$.

In PL, the diffusion parameters are defined as follows:

$$\begin{aligned} b(x) &= \lambda U'(x) - \mu, \\ a(x) &= \lambda K_a^2 U'(x) + \mu K_s^2, \end{aligned}$$

where

$$U'(x) = \begin{cases} N-1 & (0 < x \leq 1) \\ N-x & (1 < x \leq N). \end{cases}$$

Here, we propose a new approximation based on PC with the following diffusion parameters:

$$\begin{aligned} b(x) &\triangleq b_k = \lambda U''(k) - \mu, \ k-1 < x \leq k, \\ a(x) &\triangleq a_k = \lambda K_a^2 U''(k) + \mu K_s^2, \\ &\quad k-1 \leq x < k, k = 1, \cdots, N, \end{aligned}$$

where $U''(x) = N-k+1, k-1 \leq x < k, k = 1, \cdots, N$.

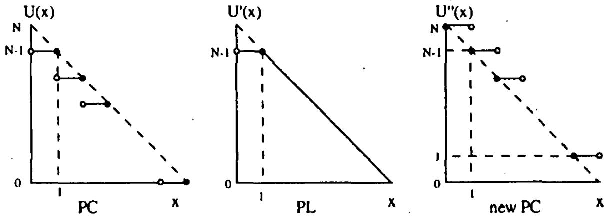

Figure 1: $U(x)$, $U'(x)$, and $U''(x)$

Table 1: The mean number of customers in the M/M/1//N sytem ($\lambda = 0.1$ and $\mu = 1$)

N	Exact	PC	PL	new PC
4	0.467	0.430	0.479	0.498
6	0.845	0.745	0.852	0.879
8	1.383	1.184	1.382	1.414
10	2.146	1.810	2.139	2.172
12	3.197	2.694	3.190	3.220
14	4.568	3.888	4.564	4.587
20	10.019	9.048	10.019	10.025

To show the difference between each approximation, we depict $U(x)$, $U'(x)$, and $U''(x)$ in Fig. 1. In addition, we introduce $V(x)$ from an asymptotic behavior of the superposition of N renewal processes, see Ref. [4]. By using $V(x)$, the diffusion parameter $a(x)$ is modified as follows:

$$a(x) \stackrel{\triangle}{=} a_k = \lambda V(k)U''(k) + \mu K_s^2, \quad k-1 < x \leq k, k = 1, \cdots, N,$$

where $V(k) = 1 + (\lambda K_a^2 - 1)/N$.

The steady-state distribution of the number of customers in the system, $\{p_k\}$, is obtained by the following equation.

$$p_k = f(k)$$

4 Numeriacl examples

We consider some special examples to examine the accuracy of the proposed approximation.

Three types of the system are considered, M/M/1//N, E_2/E_2/1//N, and H_2/H_2/1//N. Table 1, 2, and 3 show the approximation of the mean number of customers in each sytesm respectively. From these examples, the following charactersitiecs are observed:

- As for M/M/1//N, PL is the best approximation. The new PC is much better than PC, and shows similar results to PL.

Table 2: The mean number of customers in the $E_2/E_2/1//N$sytem ($\lambda = 0.1, \mu = 1, \text{and} Ka^2 = Ks^2 = 0.5$)

N	Simulation	PC	PL	new PC	new PC with $V(x)$
4	0.443	0.376	0.393	0.426	0.433
6	0.782	0.612	0.663	0.714	0.752
8	1.263	0.947	1.062	1.141	1.229
10	1.96	1.47	1.70	1.83	1.95
12	2.99	2.33	2.73	2.91	3.01
14	4.39	3.63	4.22	4.44	4.43
20	9.74	9.26	10.00	10.26	10.03

Table 3: The mean number of customers in the $H_2/H_2/1//N$ sytem ($\lambda = 0.1, \mu = 1, \text{and} Ka^2 = Ks^2 = 4$)

N	Simulation	PC	PL	new PC	new PC with $V(x)$
4	0.539	0.507	0.596	0.538	0.576
6	1.02	1.00	1.23	1.09	1.16
8	1.67	1.62	2.03	1.79	1.89
10	2.53	2.38	3.01	2.67	2.78
12	3.63	3.30	4.16	3.64	3.85
14	4.96	4.37	5.48	4.81	5.12
20	10.1	8.5	10.3	9.2	10.0

- As for E_2/E_2/1//N, H_2/H_2/1//N, the new PC is best when traffic is not so heavy. Under heavy traffic, PL shows better results than new PC.
- When N is large, new PC with $V(k)$ is most effective.

5 Conclusion

In this paper, we proposed a new diffusion approximation for the GI/GI/1//N system and showed the effectiveness of this proposal by some examples.

References

[1] L. Kleinrock, "Queieng Systems Vol. II", John Wiley & Sons (1976).

[2] T. Kimura, "A unifying diffusion model for the state dependent queue", Optimization, Vol. 18, pp. 265-283 (1987).

[3] T. Hoshiai, "Modified Diffusion Approximation for the Delay Single Server Queueing Model with Finite Srouce", IEICE Trans. B-I, Vol. J78-B-I No. 2, pp. 131-133 (1995).

[4] W, Whitt, "Approximating a point process by a renewal process I: Two basic methods", Oper. Res., 30, 1, pp. 125-147 (1982).