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Pricing of equity swaps in a stochastic interest rate economy

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1 Introduction

A swap is an agreement between two parties to exchange cash flows in the future according to a prearranged formula. The most common types of swaps include interest-rate swaps and currency swaps. Among them, equity swaps are also common in a variety of situations for practitioners, where one party, A say, agrees to pay to the other party, B say, cashflows equal to the return on a stock or equity index for a number of years and, at the same time, party B promises to make either a fixed payment or the return on another stock or equity index. Equity swaps are classified into two categories, i.e. with a *constant* notional principal and with a *variable* notional principal. In the former case, the notional principal is constant through the life of the contract, while in the latter case it changes according to the referenced equity price.

It is well known that, in the plain vanilla interest-rate swap, the swap rate is determined only through the current term structure of interest rates. Using the same argument, it can be shown that the equity swap rate with a *constant* notional principal is also determined only through the current term structure of interest rates. This result is counter-intuitive and has puzzled practitioners for many years, since equity prices have no impact on the swap rate. Recently, Chance and Rich [1998] showed that this conclusion is true only when the rate is set at the start of the contract. Then, another question arises whether or not the equity swap rate, when setting the rate at the start, is indeed independent of the equity price process even in the case of *variable* notional principals. The aim of this paper is to answer this question in a stochastic interest rate economy.

2 The economy

Under the risk-neutral probability measure P^*, the time t price of the risky asset, $S(t)$, and the instantaneous forward rate at time t for date T, $f(t,T)$, evolve according to the stochastic differential equations (abbreviated by SDE's),

$$\begin{aligned}\frac{dS(t)}{S(t)} &= r(t)dt + \sigma_1(t)dz_1(t) + \sigma_2(t)dz_2(t),\\ df(t,T) &= -\gamma(t,T)a(t,T)dt + \gamma(t,T)dz_1(t),\end{aligned}$$

for $0 \le t \le T$, respectively, where $r(t) = f(t,t)$ and

$$a(t,T) = -\int_t^T \gamma(t,u)du, \quad t \le T.$$

Here $z_1(t)$ and $z_2(t)$ denote the standard Wiener processes and they are assumed to be mutually independent.

We assume that $\sigma_i(t)$, $i = 1,2$, are constant, $\sigma_i(t) = \sigma_i$ say, and that

$$\gamma(t,T) = \gamma e^{-\int_t^T \kappa(u)du}, \quad t \le T,$$

for some deterministic function $\kappa(t)$. This case reduces to the extended Vasicek model, which is written as

$$dr(t) = \left\{\kappa(t)[f(0,t) - r(t)] + \frac{\partial}{\partial t}f(0,t) + \int_0^t \gamma^2(s,t)ds\right\} dt + \gamma dz_1(t).$$

We use the forward-neutral method to derive the pricing formulas of equity swaps. Let $S_T(t)$ be the forward price of $S(t)$, i.e.

$$S_T(t) = \frac{S(t)}{v(t,T)}, \quad t \leq T.$$

Other forward prices are defined similarly. The forward-neutral measure, P^T, is the measure under which the forward price processes are martingale. Supposing that the risk-neutral measure P^* and the forward-neutral measure P^T are both found, the next result is obtained.

Theorem 1 *Let X be the payoff of a European derivative security maturing at time T. The price of the security at time t is given by*

$$C(t) = E_t^*\left[\frac{X}{B(t,T)}\right] = v(t,T)E_t^T[X],$$

for $0 \leq t \leq T$, where E_t^* and E_t^T denote the conditional expectation operators under P^* and P^T, respectively.

If the standard Wiener processes under P^T are chosen as

$$z_1^T(t) = z_1(t) - \int_0^t a(s,T)ds, \quad z_2^T(t) = z_2(t),$$

provided that $a(t,\tau) \neq a(t,T)$, then, under P^T, the forward processes $S_T(t)$ and $v_T(t,\tau)$ satisfy the SDE's,

$$\frac{dS_T(t)}{S_T(t)} = (\sigma_1 - a(t,T))dz_1^T(t) + \sigma_2 dz_2^T(t),$$

$$\frac{dv_T(t,\tau)}{v_T(t,\tau)} = (a(t,\tau) - a(t,T))dz_1^T(t),$$

for $t < T \leq \tau$, respectively. These processes are martingales under P^T.

3 Valuation of equity swaps

An equity swap with a *variable* notional principal we consider is the following. (V1) A contract starts at time t_0, $t_0 \leq 0$, with an initial notional principal, 1 say, (V2) the payment dates are t_i, $0 < t_1 < \cdots < t_m$, with the notional principal $S(t_{i-1})/S(t_0)$, and (V3) one party pays the return on the underlying equity,

$$\left\{\frac{S(t_i)}{S(t_{i-1})} - 1\right\}\frac{S(t_{i-1})}{S(t_0)} = \frac{S(t_i) - S(t_{i-1})}{S(t_0)},$$

and the counterparty pays a fixed payment $RS(t_{i-1})/S(t_0)$ at date t_i.

Using the forward-neutral method, we derive a pricing formula of the equity swap with a variable notional principal. Especially, if the swap rate, R, is set at the start of the contract, i.e. $t_0 = 0$, then we have

$$R = \frac{m}{\sum_{i=1}^m \frac{v(0,t_i)}{v(0,t_{i-1})}\zeta(0,t_{i-1},t_i)} - 1,$$

where

$$\zeta(t,T,\tau) = e^{\int_t^T (\sigma_1 - a(s,T))(a(s,\tau) - a(s,T))ds},$$

for $t < T \leq \tau$. Note that correlation between the interest rates and the equity price process is implicit through the term $\zeta(t,T,\tau)$, whence the equity price *does* affect the equity swap rate via the correlation.

The valuation of capped equity swaps will be shown in our presentation.

References

[1] Chance D., and D. Rich. "The Pricing of Equity Swaps and Swaptions." *Journal of Derivatives*, **5**, pp.19–31.