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A Poisson arrival selection problem for Gamma prior intensity with parameter $r = 2$

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1 Introduction

Bruss (1987) studied a continuous-time generalization of the so-called secretary problem which is as follows: A man has been allowed a fixed time T in which to find an apartment. Opportunities to inspect apartments occur at the epochs of a homogeneous Poisson process of unknown intensity λ. He inspects each apartment when the opportunity arises, and he must decide immediately whether to accept or not. At any epoch he is able to rank a given apartment amongst all those inspected to date, where all permutations of ranks are equally likely and independent of the Poisson process. The objective is to maximize the probability of selecting the best apartment from those (if any) available in the interval $[0, T]$. Bruss (1987) showed that if the prior density of the intensity of the Poisson process is exponential with parameter $a > 0$, $E(1/a)$ (note that this is Gamma distribution with parameters 1 and $1/a$, $G(1, 1/a)$), then the optimal strategy is to accept the first relatively best option (if any) after time $s^* = (T + a)/e - a$.

To find the optimal strategy for Bruss's problem, Bruss directly calculated the maximum probability of selecting the best apartment when the current relatively best option is accepted, and the maximum probability of selecting the best one when the current relatively best option is rejected. In Section 2 of this paper, his problem is resolved from a different approach and it is shown that it is monotone in the sense of Chow, Robbins, and Siegmund. The following questions naturally arise: if the prior density of the intensity is Gamma $G(r, 1/a), r > 1$, what is the optimal strategy, and is it still a monotone problem?

In Section 3, the problem of Gamma prior intensity with the parameter $r = 2$ is studied in detail and the optimal strategy for the problem is solved.

2 Resolution of Bruss's problem

The one-stage look-ahead stopping strategy is employed to resolve Bruss's problem. Let $S_1, S_2, \cdots$ denote the arrival times of the Poisson process, $\{N(t)\}_{t \geq 0}$. For unknown intensity λ, an exponential prior density $a \exp\{-a\lambda\} I(\lambda > 0)$ is assumed, where a is known and nonnegative. Then by Bayes' theorem, the conditional posterior density $f(\lambda | S_j = s)$ given $S_j = s$, is $f(\lambda | S_j = s) = (\lambda^j / j!)(s + a)^{j+1} \exp\{-(s + a)\lambda\} I(\lambda > 0), s \in [0, T]$. Bruss showed that the posterior distribution of $N(T)$ given $s_1, \cdots, s_j$ is equivalent to a Pascal distribution with parameters $(j, (s + a)/(T + a))$. Using this property on $N(T)$, he obatined the following theorem.

Theorem 1 (Bruss(1987)). If the prior density of the intensity of the Poisson process is exponential with parameter $a > 0$, then the problem is monotone and the optimal strategy is to accept the first relatively best option (if any) after time $s^* = (T + a)/e - a$.

3 Bruss' problem with Gamma prior intensity

Suppose that the prior density of the intensity λ of the Poisson process is Gamma with parameters $r > 0, a > 0$, $G(r, 1/a)$. Then, the density of λ is given by

$$g(\lambda) = \frac{a^r}{\Gamma(r)} e^{-a\lambda} \lambda^{r-1}.$$

The posterior density of λ given $S_1 = s_1, \cdots, S_j = s$ can be computed and turns out to be Gamma, $G(r + j, 1/(a + s))$. We can see that the posterior distribution of $N(T)$ given $S_1 = s_1, \cdots, S_j = s$ is again a Pascal distribution.

Lemma 2. The posterior distribution of $N(T)$ given $S_1 = s_1, \cdots, S_j = s (0 < s < T)$ only depends on the values of j and S_j, and is a Pascal distribution with parameters $r + j, (s + a)/(T + a)$.

For the problem with Gamma prior density of the intensity, let $U_j(s)$, $V_j(s)$ and $W_j(s)$ denote the maximum probabilities when we face the jth option, which is the relatively best one, at time s and accept it, reject it and behave optimally hereafter, respectively . Using the formula $(n + r - 1)!/n =$

T, Q は略.

COERA 値の計算　$E(i)$ を状態 i から始まるイニングの期待値とする. 各イニングは状態1より始まるので, 1イニングの期待得点は $E(1)$ となる. これより, $E(1)$ の値を計算できれば, それを9イニング倍することにより COERA 値が決まる. $E(i)$ の計算方法は, 状態 i から状態 j に移ったときの得点を $R(j,i)$ とするこのときマルコフ連鎖の First Step Analysis より,

$$E(i) = \sum_{j=1}^{24} p(j|i)\{R(j,i) + E(j)\}, \quad i = 1, ..., 24 \tag{1}$$

ここで, R, E をベクトルで表せば (略), (1) 式は, $E = QE + R$ となり, E は

$$E = (I - Q)^{-1}R \tag{2}$$

により決まる, 状態1から始まる1イニングあたりの期待得点はベクトル E の要素 $E(1)$ となり、これより COERA 値が得られる. 状態 i における期待得点 $R(i)$ は, 規則に従い求めることができる.

3　得点圏打率を加味した最適打順決定モデル (COBO)

3.1　はじめに

COBO モデルはチームの期待得点を最大にする打順を発見するためのモデルである. COERA モデルと同様に, アウトカウントとランナーの状態を想定し規則と打撃について定め, その条件のもとで run 数の定義を基に9人の打者による期待得点数の計算を行なう.

3.2　状態, 進塁の規則, 打撃と盗塁に関する確率

2.2 で定義した状態0は COBO モデルでは状態25とする. 進塁の規則, 打撃と盗塁に関する確率は COERA と同じとする.

推移確率行列　状態の推移確率行列 $P = (P_{ij}) = p(j|i)$, $i, j = 1, 2, \cdots, 25$ は規則に従い定める (略).

1試合の期待得点導出アルゴリズム　(1) i 番バッター ($i = 1, \cdots, 9$) の攻撃に関する推移確率行列を P^i とし, $P^i = P0^i + P1^i + P2^i + P3^i + P4^i$ とする. $P0, \cdots, P4$ はそれぞれ0得点,$\cdots$,4得点となる P の分解である. (2) U_0 : 0人の打者が終ったときのイニングの状態

$$U_0 = \begin{matrix} \\ 0 \\ 1 \\ \vdots \\ 20 \end{matrix} \overset{\begin{matrix} 1 & 2 & \cdots & 25 \end{matrix}}{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix}} \text{とする.}$$

(3) U_{n+1}(j 行)=U_n(j 行)P_0 + U_n($j-1$ 行)P_1 + U_n($j-2$ 行)P_2+U_n($j-3$ 行)P_3+U_n($j-4$ 行)P_4 を使い, $U_1, U_2, U_3, \cdots$ を計算し, 各段階で, スリーアウトの状態を表す U_j の25列目の総和がある一定の確率を超えたとき, そのイニングの計算を終了し, 次のイニングを (j+1) 人目より始める. (4) U_n の25列目の i 行目の要素を x_i とすると, そのイニングでの期待得点数 r は, $r = 0x_0 + 1x_1 + 2x_2 + \cdots + 20x_{20}$ により求まる. よって, 1試合の期待得点数 R は, $R = r_1 + r_2 + \cdots + r_9$ である.

1チーム9人を特定化し, 9! 通りの打順のうち1試合あたりの期待得点数が最大となる打順を最適打順とする.

4　数値計算例

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[4] その他 (略)