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On the Minimum Augmentation of an ℓ-Connected Graph to a k-Connected Graph

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1 Introduction

Let $G = (V, E)$ stand for an undirected graph. For a nonnegative integer k, a graph G is called *k-vertex-connected* (k-connected, for short) if $|V| \geq k+1$ and the deletion of any $k-1$ or fewer vertices leaves a connected graph. Given an undirected graph $G = (V, E)$ and a positive integer k, we consider the problem of augmenting a given graph G, by the smallest number of new edges to obtain a k-connected graph. We call this problem *the k-vertex-connectivity augmentation problem* (*k-VCAP*, for short). The problem of augmenting a graph by adding the smallest number of new edges to meet vertex-connectivity requirements has been extensively studied as an important subject in the network design problem, the data security problem, the graph drawing problem and others, and many efficient algorithms have been developed so far.

Currently it is known that k-VCAP for $k \in \{2, 3, 4\}$ can be solved in polynomial time ([2, 5] for $k = 2$, [4, 8] for $k = 3$, and [3] for $k = 4$), where an initial graph may not be $(k-1)$-connected. For an arbitrary integer $k > 0$, whether k-VCAP is polynomially solvable or not is still an open question (even if an initial graph is restricted to be $(k-1)$-connected). When an initial graph is $(k-1)$-connected, Jordán presented an $O(n^5)$ time approximation algorithm for k-VCAP with a general k [6, 7] such that the difference between the number of new edges added by the algorithm and the optimal value is at most $(k-2)/2$.

It, however, is an open question whether there exists a good approximation algorithm for k-VCAP if an initial graph is not $(k-1)$-connected. For arbitrary integers k and $\delta \geq 2$, we consider whether a polynomial time algorithm can make a $(k-\delta)$-connected graph k-connected by adding a set E' of new edges such that the difference between $|E'|$ and the optimal value *opt* is small, say $|E'| - opt = O(\delta k)$.

In this paper, for arbitrary $k \geq 4$ and $\ell = k - \delta$, we consider the problem of augmenting an ℓ-connected graph G by adding the smallest number of new edges in order to make G k-connected. We first present a lower bound on the number of edges that is necessary to make a given graph G k-connected, and then show that the lower bound plus $\delta(k-1)+\max\{0, (\delta-1)(\ell-3)-1\}$ edges suffices. The task of constructing such set of new edges can be done in $O(\delta(k^2n^2 + k^3n^{3/2}))$ time.

2 Preliminaries

In $G = (V, E)$, its edge set E may be denoted by $E(G)$. For a subset $V' \subseteq V$ in G, $G - V'$ denotes the subgraph induced by $V - V'$. For an edge set F with $F \cap E = \emptyset$, we denote $G = (V, E \cup F)$ by $G + F$. An edge with end vertices u and v is denoted by (u, v). A *partition* $X_1, \cdots, X_t$ of vertex set V means a family of nonempty disjoint subsets of V whose union is V, and a *subpartition* of V means a partition of a subset of V. For two disjoint subsets of vertices $X,\ Y \subset V$, we denote by $E_G(X, Y)$ the set of edges, one of whose end vertices is in X and the other is in Y, and also denote $c_G(X, Y) = |E_G(X, Y)|$. For a subset X of V, $\{v \in V - X \mid (u, v) \in E$ for some $u \in X\}$ is called the *neighbor set* of X, denoted by $\Gamma_G(X)$. Let $p(G)$ denote the number of components in G. A *disconnecting set* of G is defined as a set S of V such that $p(G-S) > p(G)$ holds and no $S' \subset S$ has this property. If G is connected and does not contain K_n, then a disconnecting set of the minimum size is called a *minimum disconnecting set*, and its size, denoted by $\kappa(G)$, is called the *vertex-connectivity* of G. On the other hand, we define $\kappa(G) = 0$ if G is not connected, and $\kappa(G) = n-1$ if G is connected and contains the complete graph K_n. For a vertex set S in G, we call the components in $G - S$ *the S-components*, and denote the family of all S-components by $\mathcal{C}(G-S)$. A set $T \subset V$ is called *tight* if $\Gamma_G(T)$ is a minimum disconnecting set in G. A tight set D is called *minimal* if no proper subset D' of D is tight. We denote the maximum number of pairwise disjoint minimal tight sets by $t(G)$.

2.1 Edge-Splitting

Given a graph $H = (V \cup \{s\}, E)$, a designated vertex $s \notin V$ and vertices $u, v \in \Gamma_H(s)$ (possibly $u = v$), we construct graph $H' = (V \cup \{s\}, E')$ from H by deleting one edge from $E_H(s, u)$ and $E_H(s, v)$, respectively, and adding new one edges to $E_H(u, v)$. We say that H' is obtained from H by *splitting* (s, u) and (s, v).

A graph H with a designated vertex $s \in V(H)$, where $H - s$ is denoted by G, is called *s-basally k-connected* if $|\Gamma_G(X)| + c_H(s, X) \geq k$ holds for all sets $X \subset V$ with $V - X - \Gamma_G(X) \neq \emptyset$. We say that given an s-basally k-connected graph $H = (V \cup \{s\}, E)$, a pair $\{(s, u), (s, v)\}$ of two edges in $E_H(s)$ is called *k-feasible*, if the graph H' resulting from splitting edges (s, u) and (s, v) is also s-

basally k-connected. The conditions which admit k-feasible splittings at s in $(k-1)$-connected graph are given in [1]. We show the following theorem.

Theorem 2.1 *Let $H = (V \cup \{s\}, E)$ be an s-basally k-connected graph with a designated vertex s such that $G = H - s$ satisfies $\kappa(G) = k_1 - 1$ with $k > k_1 \geq 2$. Let $H' = (V \cup \{s\}, E')$ be an s-basally k_1-connected graph with $E' \subseteq E$ such that E' is minimal subject to this property. Then if $t(G) \geq \max\{k_1 + 2, 2k_1 - 2\}$, then a pair $\{(s,u),(s,w)\}$ which is k_1-feasible in H' is k-feasible in H.* □

3 An Algorithm for k-VCAP

We here outline a polynomial time algorithm for k-VCAP for an ℓ-connected input graph with $k - \ell \geq 2$ and $k \geq 4$, denoted by V-AUGMENT.

We first consider the lower bounds on the optimal value $opt_k(G)$ of the k-VCAP in G. Let $\beta_k(G) = \max\{p(G - S) \mid S \subset V$ with $|S| = k - 1\}$. To make a graph G k-connected, it is necessary to add at least $k - |\Gamma_G(X)|$ edges to $E_G(X, V - X - \Gamma_G(X))$ for each set X with $V - X - \Gamma_G(X) \neq \emptyset$ and to add at least $p(G-S)-1$ edges to connect components of $G - S$ for each set S with $|S| = k - 1$ in G.

Lower Bound: $\gamma_k(G) \equiv \max\{\lceil \alpha_k(G)/2 \rceil, \beta_k(G) - 1\}$, where $\alpha_k(G) = \max\{\sum_{i=1}^{p}(k - |\Gamma_G(X_i)|)\}$ and the max is taken over all subpartitions $\{X_1, \cdots, X_p\}$ of V such that $V - X_i - \Gamma_G(X_i) \neq \emptyset$, $i = 1, \cdots, p$. □

Now the following lemma for augmenting the vertex-connectivity is given in [6].

Lemma 3.1 [6] *Let $G = (V, E)$ be a graph and T^* be a set of vertices $v \in V$ such that every minimal tight set D satisfies $D \cap T^* \neq \emptyset$ and no $T' \subset T^*$ satisfies this property. Then G can be made $(\kappa(G) + 1)$-connected by adding at most $|T^*| - 1$ new edges.* □

Algorithm V-AUGMENT

Input: An undirected graph $G = (V, E)$ and integers $k \geq 4$ and $\ell \geq 0$ such that $|V| \geq k + 1$, $\kappa(G) = \ell$ and $k - \ell \geq 2$.

Output: A set E' of new edges with $|E'| \leq opt_k(G) + \delta(k-1) + \max\{0, (\delta-1)(\ell-3)-1\}$ such that $G^* = G + E'$ is k-connected, where $\delta = k - \ell$.

Step I. (Adding vertex s and associated edges): Add a new vertex s together with a set F^* of edges between s and V such that the resulting graph $H^* = (V \cup \{s\}, E \cup F^*)$ is s-basally k-connected and F^* is minimal subject to this property.

If $|\Gamma_{H^*}(s)| \leq k$, then halt after finding a set E' of at most $\delta(k-1)$ new edges such that $G+E'$ is k-connected by applying Lemma 3.1 δ times.

If $|\Gamma_{H^*}(s)| \geq k+1$, then $|F^*| = \alpha_k(G)$ holds and we go to Step II.

Step II. (For $(k-1)$-connectivity): Let $j := \ell$ and $H_\ell := H^*$. For $j = \ell, \ldots, k-2$, we repeat the following procedure.

As long as $t(H_j - s) \geq \max\{2j, j+3\}$, we execute a k-feasible splitting at s in Theorem 2.1. If $t(H_j - s) \leq \max\{2j-1, j+2\}$ holds, then we can add a set $\tilde{E}_j$ of at most $\max\{2j-2, j+1\}$ new edges by Lemma 3.1 and obtain an s-basally k-connected graph H_{j+1} with $\kappa(H_{j+1} - s) = j + 1$.

Step III. (For k-connectivity): We have H_{k-1} with $\kappa(H_{k-1} - s) = k - 1$. Then by executing resplitting or removing of edges incident to s in addition to splitting, we obtain the graph H_k with $G_k = H_k - s$ such that (i) we have $t(G_k) \leq \max\{2k-3, k+1\}$, or (ii) G_k can be made k-connected by adding $\beta_k(G) - 1 - |E(G_k) - E|$. In the case of (i), we can add a set $\tilde{E}_{k-1}$ of at most $\max\{2k-4, k\}$ new edges by Lemma 3.1 to obtain a k-connected $G_k + \tilde{E}_{k-1}$. We see $|E(G_k + \tilde{E}_{k-1}) - E| \leq \lceil \alpha_k(G)/2 \rceil + \delta(k-1) + \max\{0, (\delta-1)(\ell-3) - 1\}$. In the case of (ii), we obtain an optimal solution E' with $|E'| = \beta_k(G) - 1$. □

Theorem 3.1 *For a graph G with $\kappa(G) = \ell$ and a integer $k \geq 4$ with $k \geq \ell$ and $\delta = k - \ell$, G can be made k-connected by adding at most $\gamma_k(G) + \delta(k-1) + \max\{0, (\delta-1)(\ell-3)-1\}$ new edges in polynomial time.* □

References

[1] J. Cheriyan and R. Thurimella, *Fast algorithms for k-shredders and k-node connectivity augmentation*, J. Algorithms, Vol.33, 1999, pp. 15–50.

[2] K. P. Eswaran and R. E. Tarjan, *Augmentation problems*, SIAM J. Comput., Vol.5, 1976, pp. 653–665.

[3] T. Hsu, *Undirected vertex-connectivity structure and smallest four-vertex-connectivity augmentation*, Lecture Notes in Comput. Sci., 1004, Springer-Verlag, Algorithms and Computation, 1995, pp. 274–283.

[4] T. Hsu and V. Ramachandran, *A linear time algorithm for triconnectivity augmentation*, Proc. 32nd IEEE Symp. Found. Comp. Sci., 1991, pp. 548–559.

[5] T. Hsu and V. Ramachandran, *Finding a smallest augmentation to biconnect a graph*, SIAM J. Comput., Vol. 22, 1993, pp. 889–912.

[6] T. Jordán, *On the optimal vertex-connectivity augmentation*, J. Combin. Theory Ser. B, Vol.63, 1995, pp. 8–20.

[7] T. Jordán, *A note on the vertex-connectivity augmentation problem*, J. Combin. Theory Ser. B, Vol.71, 1997, pp. 294–301.

[8] T. Watanabe and A. Nakamura, *A minimum 3-connectivity augmentation of a graph*, J. Comput. System Sci., Vol.46, 1993, pp. 91–128.