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A binary search algorithm for the generalized maximum balanced flow problem

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Abstract
We consider the *generalized maximum balanced flow problem* (GMBF), i.e., the problem of finding a *generalized maximum flow* in a network such that each arc-flow value is bounded by a given fixed proportion of the total flow value of a generalized flow, and propose a polynomial algorithm for this problem. Problem (GMBF) can be regarded as a generalization of the *maximum balanced flow problem* (MBF) for which several efficient algorithms have been proposed by Minoux and Zimmermann, etc.

1. Problem (GMBF)

Let $G = (V, A)$ be a directed graph with vertex set V and arc set A, where $n \equiv |V|$ and $m \equiv |A| \geq 2$. $s \in V$ (resp. $t \in V$) is given *source* (resp. *sink*). $\partial^+ a$ (resp. $\partial^- a$) $(a \in A)$ is *tail* (resp. *head*) of $a \in A$. $\delta^{\pm} v \equiv \{a \in A : \partial^{\pm} a = v\}$ for $v \in V$. Let $u(a)$ be an integral *capacity* of $a \in A$. For each $a \in A$, $\gamma(a)$ is a rational *gain* and $\alpha(a)$ is a rational *balancing rate*, where each of them is expressed as a ratio of two integers $\frac{\gamma_0(a)}{\gamma_1(a)}$ or $\frac{\alpha_0(a)}{\alpha_1(a)}$. Given a network $N = (G, u, \gamma, \alpha, \beta, s, t)$, problem (GMBF) is defined as follws:

(GMBF) : Maximize $\mathrm{val}_N(f)$ s.t.

$$\partial_\gamma f(v) = 0, \ (v \in V - \{s, t\}), \quad (2.1)$$
$$0 \leq f(a) \leq u(a), \ (a \in A), \quad (2.2)$$
$$f(a) \leq \alpha(a)\mathrm{val}_N(f) + \beta(a), \quad (a \in A), \quad (2.3)$$

where integers $\beta(a)$ $(a \in A)$ are given, and $\mathrm{val}_N(f) = \sum_{a \in \delta^-(t)} \gamma(a) f(a)$ and $\partial_\gamma f(v) \equiv \sum_{a \in \delta^+(v)} f(a) - \sum_{a \in \delta^-(v)} \gamma(a) f(a)$. Given a network $N' = (G, u, \gamma, s, t)$, problem (GMF) is:

(GMF): Maximize $\mathrm{val}_{N'}(f)$ s.t. (2.1)$\sim$(2.2).

Given a network N_z, i.e., N with a parameter $z \geq 0$, consider problem (GMF(z)), where f in (2.1) and the inequalities in (2.2) should be replaced by f_z and

$$0 \leq f_z(a) \leq u_z(a), \ (a \in A), \quad (2.4)$$

where $u_z(a) = \min\{u(a), \alpha(a)z + \beta(a)\}$. A flow f of N is a funcion $f : A \to \mathbf{R}_+$ satisfying (2.1) $\sim$ (2.3), where $\mathbf{R}_+$ is the set of nonnegative reals. A flow f_z of N_z is a funcion $f_z : A \to \mathbf{R}_+$ satisfying (2.1) and (2.4). The value of a flow f of N is $\mathrm{val}_N(f)$. An optimal flow f of N (resp. f_z of N_z) is a flow f of N (resp. f_z of N_z) maximizing the value of f (resp. f_z). For a flow f_z of N_z, define $A(f_z) \equiv A^+(f_z) \cup A^-(f_z)$,

$$u_z^{f_z}(a) = \begin{cases} u_z(a) - f_z(a) & (a \in A^+(f_z)), \\ \gamma(\bar{a}) f_z(\bar{a}) & (a \in A^-(f_z)), \end{cases}$$

$$\gamma^{f_z}(a) = \begin{cases} \gamma(a) & (a \in A^+(f_z)), \\ 1/\gamma(\bar{a}) & (a \in A^-(f_z)), \end{cases}$$

where $A^+(f_z) = \{a \in A : f_z(a) < u_z(a)\}$, $A^-(f_z) = \{a \equiv (j, i) : \bar{a} \equiv (i, j) \in A, f_z(\bar{a}) > 0\}$. A *residual* network w.r.t. a flow f_z of N_z is defined as $N_z(f_z) = (G(f_z) = (V, A(f_z)), u_z^{f_z}, \gamma^{f_z}, s, t)$. The dual (DGMF($z$)) for (GMF($z$)) is

(DGMF(z)): Minimize $\sum_{a \in A} u_z(a) \theta_z(a)$ s.t.
$\bar{\gamma}^{\pi_z}(a) + \theta_z(a) \geq 0, (a \in A)$,
$\theta_z(a) \geq 0, (a \in A)$,
$\pi_z(v)$: a real, $(v \in V)$,

where $\pi_z(s) = 0, \pi_z(t) = 1$, and $\bar{\gamma}^{\pi_z}(a) = \pi_z(\partial^+ a) - \gamma(a)\pi_z(\partial^- a)$. Define $\theta_z(a) \equiv \max\{0, -\bar{\gamma}^{\pi_z}(a)\}$. Complementary slackness conditions imply that if $f_z(a) > 0$ (resp. $f_z(a) < u_z(a)$) for each $a \in A$,then $\bar{\gamma}^{\pi_z}(a) + \theta_z(a) = 0$ (resp. $\theta_z(a) = 0$). Define $A_z \equiv \{a \in A : u(a) > \alpha(a)z + \beta(a)\}$, $C(\theta_z) \equiv \sum_{a \in A_z} \alpha(a)\theta_z(a)$,and $D(\theta_z) \equiv \sum_{a \in A_z} \beta(a)\theta_z(a) + \sum_{a \in A - A_z} u(a)\theta_z(a)$. The optimal value of (DGMF(z)) is expressed as $\mathrm{val}_{N_z}(f_z^*) = C(\theta_z^*)z + D(\theta_z^*)$, where f_z^* is optimal in N_z and θ_z^* is optimal for (DGMF(z)).

Proposition 1
For an optimal flow f_z^* of N_z,a function $F(z) = \mathrm{val}_{N_z}(f_z^*)$ is nondecreasing, continuous, piecewise

linear, and concave. □

Proposition 2
The value z^* of an optimal flow in N is

$$z^* = \begin{cases} \max\{z : z = \frac{D(\theta_z^*)}{1-C(\theta_z^*)}\}, & (D(\theta_z^*) \neq 1), \\ \max\{z : z = \mathrm{val}_{N_z}(f_z^*)\}, & (D(\theta_z^*) = 1), \end{cases}$$

where θ_z^* is a dual optimal solution. □

Let P_v be a directed path from $v \in V$ to t in N'. The *gain* of P_v w.r.t. γ is $\gamma(P_v) \equiv \Pi_{a\in P_v}\gamma(a)$. The highest gain path from v to t is $\max \gamma(P_v)$. A *labeling* function w.r.t. N' is a function $\mu : V \to \mathbf{R}_{++} \cup \{\infty\}$ such that $\mu(t) = 1$ where $\mathbf{R}_{++} \equiv \{r \in \mathbf{R}_+ : r > 0\}$. The *relabeled gain* of $a \in A$ w.r.t. γ and μ is defined by $\gamma_\mu(a) = \gamma(a)\mu(\partial^+ a)/\mu(\partial^- a)$. The *canonical label* of $v \in V$ in N' is the inverse of the highest gain path from v to t. If no such path exists, its label is ∞.

Theorem 3[Wayne,1999]
A flow g of N' is maximum if and only if there exists a labeling function μ such that:

$\gamma_\mu(a) \leq 1, \quad (a \in N'(g)),$

$\mu(v) = \infty, \quad (v \notin T'),$

where $N'(g)$ is the residual network w.r.t. g, and T' is the set of vertices reachable to t by using arcs in $N'(g)$. □

2. Binary Search Algorithm

Input: $N = (G, u, \gamma, \alpha, \beta, s, t)$
Output: an optimal flow if it exists
Step 1: Initialization
  Set $U \leftarrow mB^2$, where $B \equiv \max_{a\in A}\{\gamma_0(a), \gamma_1(a), \alpha_1(a), u(a), |\beta(a)|\}$.
  If $\mathrm{val}_{N_U}(f_U) \geq U$, **then** stop.
  Set $L \leftarrow \max\{0, \max_{a\in A} \frac{-\beta(a)}{\alpha(a)}\}$.
  If $\mathrm{val}_{N_L}(f_L) \geq L$, **then** go to step 3
  else if $C(\theta_L^*) \leq 1$, **then** stop.
Step 2: Decision of a lower bound
  Set $z_a \leftarrow \frac{u(a)-\beta(a)}{\alpha(a)}$ for each $a \in A$.
  Choose a maximal set $\{z_{a_i} : z_{a_i} > L, z_{a_i} < z_{a_{i+1}} (1 \leq i \leq m'-1)\}$ for some $m' \leq m$.
  Set $z_{a_{m'+1}} = U$, $z_{a_0} \leftarrow L$, and $i \leftarrow 0$.
 repeat
  Set $i \leftarrow i+1$.
  If $\mathrm{val}_{N_{z_{a_i}}}(f_{z_{a_i}}) \geq z_{a_i}$, **then** set $L \leftarrow z_{a_i}$ and go to step 3.
  If $C(\theta^*_{z_{a_{i-1}}}) \neq C(\theta^*_{z_{a_i}})$, **then**
  { set
   $z' \leftarrow \frac{D(\theta^*_{z_{a_i}})-D(\theta^*_{z_{a_{i-1}}})}{C(\theta^*_{z_{a_{i-1}}})-C(\theta^*_{z_{a_i}})}$.
   If $\mathrm{val}_{N_{z'}}(f_{z'}) \geq z'$, **then** set $L \leftarrow z'$ and go to step 3
   else if $C(\theta^*_{z_{a_i}}) \leq 1$, **then** stop.
  }
 until $i = m'+1$.
 If $i = m'+1$, **then** stop.
Step 3: Binary search
 repeat
  Set $z \leftarrow \frac{L+U}{2}$.
  Stop **if** $\mathrm{val}_{N_z}(f_z) = z$ and $C(\theta^*{}_z) < 1$.
  If $\mathrm{val}_N(f_z) \geq z$, **then** set $L \leftarrow z$
  else set $U \leftarrow z$.
 until $U - L < \frac{1}{B^{5m}}$.
Step 4: Decision of an optimal flow
  Set $z^* \leftarrow \frac{D(\theta_U^*)}{1-C(\theta_U^*)}$.
  Find an optimal flow f_{z^*} of N_{z^*} and stop.

Proposition 4
For any z, $D(\theta_z)$ (resp. $C(\theta_z)$) is an integral multiple of ${\Gamma_1}^{-1}$ (resp. $(\Gamma_1\Gamma_2)^{-1}$) where $\Gamma_1 \equiv \Pi_{a\in A}(\gamma_0(a)\gamma_1(a))$ and $\Gamma_2 \equiv \Pi_{a\in A}\alpha_1(a)$. □

Proposition 5
Let $[L_i, U_i]$ be the interval after ith repetition of Step 3. There exists an optimal value z'' such that $L_i \leq z'' < U_i$ and $C(\theta^*_{z''}) < 1$ if the algorithm is continuing. □

Proposition 6
Let $[L, U]$ be the interval at the beginning of step 4. There is a cornerpoint $(z', F(z'))$ with $z' \in [L, U]$, where a cornerpoint is a point $(z', F(z'))$ such that $F(z)$ is not differential at $z = z'$. □

Theorem 7
The algorithm runs in $O(m \log B\ M(n, m))$ time, where $M(n, m)$ is the complexity for solving a generalized maximum flow problem in a network with n vertices and m arcs. □

Reference
K.D. Wayne: Generalized Maximum Flow Algorithms, Ph.D. thesis, Cornell Univ., 1999.