Linear Relaxation for Hub Network Design Problems

Univ. of Tokyo Hiro-o SAITO
02601790 Univ. of Tokyo Shiro MATUURA
01605000 Univ. of Tokyo Tomomi MATSUI*

1. Introduction

We consider a network design problem with hub-and-spoke structure which arises from the airline industry[2, 3]. The hub-and-spoke structure models the situation such that some nodes, called non-hub nodes, can interact only via a set of completely interconnected nodes, called hub nodes. This paper deals with the case that the hub nodes and non-hub nodes are fixed. We consider the problem which allocates each non-hub node to one of hub nodes, and minimizes the total transportation cost. We call this problem a single allocation problem or HLP.

We propose a relaxation technique for the problem, which linearizes the non-convex quadratic objective function of the original quadratic integer programming problem.

Let H, N be the set of hub nodes and non-hub nodes, respectively. We define $|H| = h$ and $|N| = n$. Let $A \subseteq \{(p,q) \in N \times N \mid p \neq q\}$ be a given set of non-hub pairs.

For any pair of non-hub nodes $(p,q) \in A$, the flow from p to q is denoted by w_{pq}. For any pair of nodes $(i,j) \in (H \times H) \cup (H \times N) \cup (N \times H)$, the unit transportation cost from i to j is denoted by $c_{ij} \geq 0$.

We can formulate the problem HLP as follows;

$$\begin{array}{ll} \text{min.} & \sum_{(p,q)\in A} w_{pq}(\sum_{i\in H} c_{pi}x_{pi} \\ & \quad + \sum_{i\in H}\sum_{j\in H} c_{ij}x_{pi}x_{qj} + \sum_{j\in H} c_{jq}x_{qj}) \\ \text{s. t.} & \sum_{i\in H} x_{pi} = 1 \quad (\forall p \in N), \\ & x_{pi} \in \{0,1\} \quad (\forall (p,i) \in N \times H). \end{array}$$

2. Linearization

For any pair $(p,q) \in A$, we denote the corresponding quadratic term $f_{pq}(\boldsymbol{x}) \stackrel{\text{def.}}{=} \sum_{i\in H}\sum_{j\in H} c_{ij}x_{pi}x_{qj}$ where $\boldsymbol{x} = (x_{p1},\ldots,x_{ph}; x_{q1},\ldots,x_{qh})^\top$. We denote the sets of indices of $\boldsymbol{x}$ by P, Q, i.e., $P \stackrel{\text{def.}}{=} \{p1,\ldots,ph\}$ and $Q \stackrel{\text{def.}}{=} \{q1,\ldots,qh\}$. We define the sets Ω_{pq} and $\mathcal{H}_{pq}$ by

$$\begin{array}{rcl} \Omega_{pq} & \stackrel{\text{def.}}{=} & \{\boldsymbol{x} \in \{0,1\}^P \times \{0,1\}^Q \mid \\ & & \sum_{i\in H} x_{pi} = 1,\ \sum_{i\in H} x_{qi} = 1\}, \\ \mathcal{H}_{pq} & \stackrel{\text{def.}}{=} & \{(f_{pq}(\boldsymbol{x}), \boldsymbol{x}) \mid \boldsymbol{x} \in \Omega_{pq}\}, \end{array}$$

respectively. Throughout this paper, conv$S$ denotes the convex hull of S.

Our approach is to replace the function $f_{pq}(\boldsymbol{x})$ by the lower hull of the set $\mathcal{H}_{pq}$. We define the function $\underline{f}_{pq} : \text{conv}\Omega_{pq} \to \mathrm{R}$ by $\underline{f}_{pq}(\boldsymbol{x}) = \min\{z \mid (z,\boldsymbol{x}) \in \text{conv}\mathcal{H}_{pq}\}$.

In the following, we show that the above function has a relation to Hitchcock transportation problems. Let $B_{pq} \stackrel{\text{def.}}{=} (P,Q;E)$ be the complete bipartite graph with vertex sets P, Q and edge set $E = P \times Q$. For each edge $(i,j) \in P \times Q$, we associate the cost c_{ij}. Given a nonnegative vector $\boldsymbol{x} \in \mathrm{R}^P \times \mathrm{R}^Q$ we define the following linear programming problem;

$$\begin{array}{ll} \text{HTP}(\boldsymbol{x}): & \\ \text{min.} & \sum_{pi\in P}\sum_{qj\in Q} c_{ij}\lambda_{ij} \\ \text{s. t.} & \sum_{qj\in Q} \lambda_{ij} = x_{pi} \quad (\forall pi \in P), \\ & \sum_{pj\in P} \lambda_{ij} = x_{qj} \quad (\forall qj \in Q), \\ & \lambda_{ij} \geq 0 \quad (\forall (pi,qj) \in P \times Q). \end{array}$$

The above problem is called *Hitchcock transportation problem.*

Now, we describe our main result.

Theorem For any vector $\boldsymbol{x} \in \mathrm{conv}\Omega_{pq}$, $\underline{f}_{pq}(\boldsymbol{x})$ is equivalent to the optimal value of the linear programming problem HTP($\boldsymbol{x}$).

The above theorem shows that HLP can be transformed to the following integer programming problem;

P1 :

$$\begin{array}{ll} \text{min.} & \sum_{(p,q)\in A} w_{pq}(\sum_{i\in H} c_{pi}x_{pi} \\ & + \sum_{i\in H}\sum_{j\in H} c_{ij}\lambda_{pi,qj} + \sum_{j\in H} c_{jq}x_{qj}) \\ \text{s. t.} & \sum_{i\in H} x_{pi} = 1 \quad (\forall p \in N), \\ & x_{pi} \in \{0,1\} \quad (\forall (p,i) \in N \times H), \\ & \sum_{j\in H} \lambda_{pi,qj} = x_{pi} \quad (\forall (p,q,i) \in A \times H), \\ & \sum_{i\in H} \lambda_{pi,qj} = x_{qj} \quad (\forall (p,q,i) \in A \times H), \\ & \lambda_{pi,qj} \in \{0,1\} \quad (\forall (p,q,i,j) \in A \times H^2). \end{array}$$

3. Dual Transportation Polyhedra

Here, we discuss the explicit representation of the function $\underline{f}_{pq}(\boldsymbol{x})$ as a piecewise linear convex function. A vector $\boldsymbol{y} \in \mathrm{R}^P \times \mathrm{R}^Q$ is called *feasible* when $y_{pi}+y_{qj} \leq c_{ij}$ for all $(pi,qj) \in P \times Q$. It is clear that for any feasible vector $\boldsymbol{y} \in \mathrm{R}^P \times \mathrm{R}^Q$, the linear function $g(\boldsymbol{x}) = \boldsymbol{y}^\top \boldsymbol{x}$ satisfies that $g(\boldsymbol{x}) \leq f_{pq}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega_{pq}$. Given a spanning tree T of B, we denote the vector $\boldsymbol{y} \in \mathrm{R}^P \times \mathrm{R}^Q$ satisfying the conditions that $y_{p1} = 0$ and $y_{pi} + y_{qj} = c_{ij}$ for each edge (pi,qj) in the spanning tree T by $\boldsymbol{y}(T)$. It is well-known that for any spanning tree T of B_{pq}, the vector $\boldsymbol{y}(T)$ is uniquely defined. A spanning tree T of B_{pq} is called *feasible* when $\boldsymbol{y}(T)$ is a feasible vector. We denote the set of all the feasible spanning trees by $\mathcal{T}_{pq}$. Then we have the following theorem.

Theorem For any vector $\boldsymbol{x} \in \mathrm{conv}\Omega_{pq}$, $\underline{f}_{pq}(\boldsymbol{x}) = \min\{z \mid z \geq \boldsymbol{y}(T)^\top \boldsymbol{x} \ (\forall T \in \mathcal{T}_{pq})\}$.

The above theorem shows that we can transform HLP to an optimization problem defined as follows;

P2 :

$$\begin{array}{ll} \text{min.} & \sum_{(p,q)\in A} w_{pq}(\sum_{i\in H} c_{pi}x_{pi} + z_{pq} \\ & + \sum_{j\in H} c_{jq}x_{qj}) \\ \text{s. t.} & \sum_{i\in H} x_{pi} = 1 \quad (\forall p \in N), \\ & x_{pi} \in \{0,1\} \quad (\forall (p,i) \in N \times H), \\ & z_{pq} \geq \boldsymbol{y}(T)^\top \boldsymbol{x} \quad (\forall (p,q) \in A, \ \forall T \in \mathcal{T}_{pq}). \end{array}$$

The number of inequalities appearing in the definition of the function $\underline{f}_{pq}(\boldsymbol{x}) = \min\{z \mid z \geq \boldsymbol{y}(T)^\top \boldsymbol{x} \ (\forall T \in \mathcal{T}_{pq})\}$ is bounded by ${}_{2(h-1)}\mathrm{C}_{h-1}$[1].

4. Discussions

We applied our approach to the benchmark test problems provided by O'Kelly in [2]. Computer experiences were performed for the problems with 3 hub nodes chosen from set of nodes. For every problem, our linear relaxation problem found an integer optimal solution. This result indicates that our relaxation is very tight.

References

[1] M. L. Balinski and A. Russakoff, Faces of dual transportation polyhedra, *Math. Prog. Study*, 22 (1984), 1–8.

[2] M. E. O'Kelly, A quadratic integer program for the location of interacting hub facilities, *EJOR*, 32 (1987), 393–404.

[3] J. Sohn and S. Park, A linear program for the two-hub location problem, *EJOR*, 100 (1997), 617–622.