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Sealed Bid Multi-object Auctions with Necessary Bundles and its Application to Spectrum Auctions

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1 Introduction

In this paper, we consider multi-object auctions in which each bidder has a positive reservation value for only one special subset of objects, called a necessary bundle. We show that this auction leads to an efficient allocation through Nash equilibria under complete information when the bid-grid size is sufficiently small. We apply our results to spectrum auctions satisfying the conditions that necessary bundles are intervals of discretized spectrum. We show that the revenue maximization problem for the seller can be solved in polynomial time for the above auctions. The algorithm also indicates a method to choose an accepted bidder randomly when the revenue maximization problem has multiple optimal solutions. Lastly, we introduce a linear inequality system which characterizes the set of Nash equilibria.

2 The Model

Let $N=\{1,2,\ldots,n\}$ be the set of bidders, and $M=\{1,2,\ldots,m\}$ the set of objects. We assume that each bidder has a positive reservation value only for one special subset of objects, called the *necessary bundle*. If the bidder misses any object in the bundle, other objects in the bundle are not valuable to the bidder at all. We also assume that the objects in the bundle are also sufficient to the bidder, so the bidder has no value for any object out of the necessary bundle. We denote the necessary bundle of the bidder i by T_i and its value for T_i by $v_i>0$. The reservation value $V_i(S)$ for any $S\subseteq M$ is defined by

$$V_i(S)=\begin{cases} v_i & (T\subseteq S),\\ 0 & (\text{otherwise}).\end{cases}$$

Throughout this paper, we assume that $v_i\in\{\delta,2\delta,3\delta,\ldots\}$ for any $i\in N$.

Next, we propose the sealed bid simultaneous auctions with necessary bundles. At the beginning of the auction, each bidder $i\in N$ submits a bid (B_i,b_i) where B_i is a bundle and the nonnegative real number b_i is the amount it is willing to pay for the bundle B_i. We assume that each bid a multiple of $\varepsilon=\frac{\delta}{I}$. for some integer I. The set of integer multiples of the bid unit is denoted by Z_ε. In the following, we write a profile of bids $((B_1,b_1),(B_2,b_2),\ldots,(B_n,b_n))$ as $(\boldsymbol{B},\boldsymbol{b})$ by changing the order of components where $\boldsymbol{B}=(B_1,B_2,\ldots,B_n)$ and $\boldsymbol{b}=(b_1,b_2,\ldots,b_n)$.

The seller solves the following integer programming problem, called the Bundle Assignment Problem (BAP), which maximizes the revenue:

$$\begin{array}{lll}\text{BAP}(\boldsymbol{B},\boldsymbol{b}): & & \\ \quad\text{maximize} & \displaystyle\sum_{i\in N} b_i x_i=\boldsymbol{b}\cdot\boldsymbol{x} & \\ \quad\text{subject to} & \displaystyle\sum_{i:B_i\ni j} x_i\leq 1 & (\forall j\in M),\\ & x_i\in\{0,1\} & (\forall i\in N),\end{array}$$

where $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)$. We denote the set of all the optimal solutions of $\text{BAP}(\boldsymbol{B},\boldsymbol{b})$ by $\Omega(\boldsymbol{B},\boldsymbol{b})$. The seller solves the problem $\text{BAP}(\boldsymbol{B},\boldsymbol{b})$ and obtains an optimal solution $\boldsymbol{x}^*$. If the problem $\text{BAP}(\boldsymbol{B},\boldsymbol{b})$ has multiple optimal solutions, the seller chooses an optimal solution $\boldsymbol{x}^*\in\Omega(\boldsymbol{B},\boldsymbol{b})$ at random. Hence, for the given profile $(\boldsymbol{B},\boldsymbol{b})$, the expected utility of the bidder i $U_i(\boldsymbol{B},\boldsymbol{b})$ is defined as follows:

$$U_i(\boldsymbol{B},\boldsymbol{b})\stackrel{\text{def.}}{=}\begin{cases} V_i(B_i)-b_i & (i\in \text{P}(\boldsymbol{B},\boldsymbol{b})),\\ (V_i(B_i)-b_i)\dfrac{|\{\boldsymbol{x}\in\Omega(\boldsymbol{B},\boldsymbol{b})\mid x_i=1\}|}{|\Omega(\boldsymbol{B},\boldsymbol{b})|} & \\ \qquad\qquad (i\in \text{Q}(\boldsymbol{B},\boldsymbol{b})), & \\ 0 & (i\in \text{R}(\boldsymbol{B},\boldsymbol{b})),\end{cases}$$

where

$$\begin{array}{lll}\text{P}(\boldsymbol{B},\boldsymbol{b}) & \stackrel{\text{def.}}{=} & \{i\in N\mid x_i=1,\ \forall\boldsymbol{x}\in\Omega(\boldsymbol{B},\boldsymbol{b})\},\\ \text{R}(\boldsymbol{B},\boldsymbol{b}) & \stackrel{\text{def.}}{=} & \{i\in N\mid x_i=0,\ \forall\boldsymbol{x}\in\Omega(\boldsymbol{B},\boldsymbol{b})\},\\ \text{Q}(\boldsymbol{B},\boldsymbol{b}) & \stackrel{\text{def.}}{=} & N\setminus(\text{P}(\boldsymbol{B},\boldsymbol{b})\cup\text{R}(\boldsymbol{B},\boldsymbol{b})).\end{array}$$

Papers [1] discuss some algorithms for solving the problem $\mathrm{BAP}(\boldsymbol{B}, \boldsymbol{b})$.

3 Pure Strategy Nash Equilibria

In this section, we discuss the existence of pure strategy Nash equilibria. We say that a profile $(\boldsymbol{B}^*, \boldsymbol{b}^*)$ is a *Nash equilibrium* when $(\boldsymbol{B}^*, \boldsymbol{b}^*)$ satisfies the conditions that for each bidder $i \in N$, $U_i(\boldsymbol{B}^*, \boldsymbol{b}^*) \geq U_i((B_i, \boldsymbol{B}^*_{-i}), (b_i, \boldsymbol{b}^*_{-i}))$ for any bid (B_i, b_i) where $B_i \subseteq M$ and $b_i \in Z_\varepsilon$.

Let us consider the following set:

$$\mathcal{F}_\varepsilon(\boldsymbol{B}, \boldsymbol{v}) \stackrel{\text{def.}}{=} \{\boldsymbol{b} \in Z_\varepsilon^N | b_i = v_i \ (\forall i \in \mathrm{R}(\boldsymbol{B}, \boldsymbol{v}) \cup \mathrm{Q}(\boldsymbol{B}, \boldsymbol{v})), b_i \leq v_i - 2^n\varepsilon \ (\forall i \in \mathrm{P}(\boldsymbol{B}, \boldsymbol{v})), \Omega(\boldsymbol{B}, \boldsymbol{b}) = \Omega(\boldsymbol{B}, \boldsymbol{v})\}$$

which is a subset of bid price vectors satisfying that the set of optimal solutions does not change. Given a subset of bid price vectors $X \subseteq Z_\varepsilon^N$, a vector $\boldsymbol{b} \in X$ is called a *minimal vector in* X if and only if for any $\boldsymbol{b}' \in Z_\varepsilon^N$, the condition $[\boldsymbol{b}' \leq \boldsymbol{b}$ and $\boldsymbol{b}' \neq \boldsymbol{b}]$ implies $\boldsymbol{b}' \notin X$.

Theorem 1 *If $\mathcal{F}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$ is non-empty, then for any minimal vector $\boldsymbol{b}^*$ in $\mathcal{F}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$, $(\boldsymbol{T}, \boldsymbol{b}^*)$ is a Nash equilibrium.*

The following theorem shows the non-emptiness of $\mathcal{F}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$.

Theorem 2 *If ε is a sufficiently small positive number, $\mathcal{F}_\varepsilon(\boldsymbol{T}, \boldsymbol{v})$ is non-empty.*

4 Spectrum Auctions

In this section, we consider a spectrum auction. An auctioneer wants to sell licenses for radio spectrum $M = \{1, 2, \cdots, m\}$. Each spectrum $j (\geq 2)$ is adjacent to $j-1$ on the right. We consider the case of the preferences in which each bidder i requires any spectrum j satisfying $\ell_i \leq j \leq h_i$ but no spectrum outside of it. This setting is applied to our model, auctions with necessary bundles, where agent i's necessary bundle is an interval $T_i = \{j \in M \mid \ell_i \leq j \leq h_i\}$.

Bundle Assignment Problem and Longest Path Problem In the spectrum auctions, we denote BAP by $\max\{\boldsymbol{b} \cdot \boldsymbol{x} \mid A\boldsymbol{x} \leq \boldsymbol{1},\ \boldsymbol{x} \in \{0,1\}^N\}$. Then the coefficient matrix $A \in \mathrm{R}^{M \times N}$ belongs to the class of consecutive one matrices. It is well-known that the consecutive one matrices are totally unimodular. And so, the linear relaxation problem of BAP, $\max\{\boldsymbol{b} \cdot \boldsymbol{x} \mid A\boldsymbol{x} \leq \boldsymbol{1},\ \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{1}\}$, has a 0-1 valued optimal extreme point solution for any vector $\boldsymbol{b}$. Thus, when we solve the above linear programming problem by polynomial time method, we can assert the following proposition.

Proposition 1 *In the spectrum auctions, the related bundle assignment problem (BAP) is polynomially solvable.*

Moreover, we can show that the above problem is essentially equivalent to the longest path problem and we can solve the problem by CPM (critical path method) for PERT (Program Evaluation and Review Technique). The critical path method, which is equivalent to the ordinary dynamic programming technique, finds a longest path from 0 to m in G in $\mathrm{O}(n+m)$ time.

Random Selection from Multiple Optimal Solutions When BAP has multiple optimal solutions, the seller needs to choose an optimal solution at random. In general, it is hard to enumerate all optimal solutions, but by using the equivalence between the longest path problems and spectrum auctions, we can obtain the following proposition.

Proposition 2 *When BAP has multiple optimal solutions in a spectrum auction, an optimal solution can be randomly choosed by a polynomial time method.*

References

[1] Rothkopf, M. H., Pekeć, A. and Harstad, R. M. (1998), "Computationally manageable combinatorial auctions", *Management Science*, vol. 44, 1131–1147.