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A Search Game on a Cyclic Graph*

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1. Introduction.

Let us imagine a system which can be regarded as a network, when we must find some parts which cause the breakdown of the system, or in a building, when we must patrol and find someone which threatens the security. In these cases we must decide how to move and examine in order to find the object with less time and cost. This problem is modeled as a search problem on a finite graph, in which the nodes are examined and the seeker moves along the edges.

The purpose of this report is to analyze search problems with traveling cost on a finite graph, based on the minimax decision rule, when the graph is a cyclic graph. We model the problem as a game between the hider and the seeker.

2. The Model and Notation.

A (undirected) *graph* is an ordered pair (V, E) in which V is a finite set of *nodes*, labeled from 0 to n, and E is a finite set of pairs of nodes, (i,j), called *edges*. Thus, $V \equiv \{0,1,\ldots,n\}$ and E is a subset of $\{(i,j) : i,j \in V\}$. If $(i,j) \in E$, we say i and j are *adjacent* nodes. A *path* between i_0 and i_s is an ordered $(s+1)$-tuple $\pi = (i_0, i_1, \ldots, i_s)$ such that $(i_{r-1}, i_r) \in E$ for $r = 1, \ldots, s$. Each edge $(i,j) \in E$ is associated with a positive real number $d(i,j) > 0$, called the *weight* of $(i,j) \in E$. The *length* of a path is the sum of the weights of the edges in the path. If $i,j \in V$ and $(i,j) \notin E$, we define $d(i,j)$ by the minimum of the lengthes of the pathes between i and j.

In this report we assume $G = (V, E)$ is a *cyclic* graph, that is, for $0 < i < n$, i is adjacent only to $i-1$ and $i+1$, and n is adjacent to 0 and $n-1$.

Player 1 (called the hider) hides among one of all nodes except for the node 0, and stays there. Player 2 (called the seeker) examines each node until he finds the hider, traveling along edges. We assume that at the beginning of the search the seeker is at 0, and that he chooses a path which minimizes the length between i and j when $(i,j) \notin E$ and he examines i after having examined j. Associated with the examination of i $(1 \le i \le n)$ is the examination cost that consists of two parts:(i) a traveling cost $d(i,j) > 0$ of examining i after having examined j, and (ii) an examination cost $c(i) > 0$. For simplicity we assume

$$d(i,j) = 1 \text{ for all } (i,j) \in E. \tag{1}$$

There is not a probability of overlooking the hider, given that the right node is examined. For convenience we let $d(i,i) = 0$ for all $i \in V$. Before searching (hiding resp.), the seeker (the hider) must determine a strategy so as to make the cost of finding the hider as small (large resp.) as possible.

A (pure) *strategy* for the hider is expressed by an element, say i, of $V \backslash \{0\}$, which means the hider determines on hiding in i. Let Σ be the set of all permutations on $V \backslash \{0\}$. A strategy for the seeker is an element σ in Σ, which means the seeker examines $\sigma(1), \ldots, \sigma(n)$ in this

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order. For convenience we let $\sigma(n+1) = \sigma(0) = 0$. We define $\sigma^* \in \Sigma$ by $\sigma^*(i) = i$ for all $i \in V\backslash\{0\}$. For any $\sigma \in \Sigma$, $r\sigma \in \Sigma$ is defined by $r\sigma(i) = \sigma(n-i+1)$ for $i \in V\backslash\{0\}$.

For a strategy pair $(i,\sigma) \in V\backslash\{0\} \times \Sigma$, the cost of finding the hider, written as $f(i,\sigma)$, is:

$$f(i,\sigma) = \sum_{x=1}^{\sigma^{-1}(i)} \{d(\sigma(x),\sigma(x-1)) + c(\sigma(x))\}. \tag{2}$$

Letting payoffs for the hider and the seeker be $f(i,\sigma)$ and $-f(i,\sigma)$ respectively, we have a two-person zero-sum (finite) game $(V,\vec{c})$, where $\vec{c} = (c(1),\ldots,c(n))$. We denote the value of this game by $v \equiv v(\vec{c}) \equiv v(V,\vec{c})$. Then we consider the mixed extension of the game in a usual way. $p = (p(1),\ldots,p(n))$ is a (mixed) strategy of the hider, where $p(i)$ is the probability that the hider chooses $i \in V\backslash\{0\}$. $q = \{q(\sigma)\}$ is a strategy of the seeker, where $q(\sigma)$ is the probability that the seeker chooses $\sigma \in \Sigma$. P and Q are strategy spaces of the hider and the seeker respectively. Thus,for $p \in P$ and $q \in Q$,

$$\sum_{i=1}^{n} p(i) = 1, p(i) \geq 0 \text{ for all } i \in V\backslash\{0\}, \text{ and } \sum_{\sigma\in\Sigma} q(\sigma) = 1, q(\sigma) \geq 0 \text{ for all } \sigma \in \Sigma. \tag{3}$$

For $(p,q) \in P \times Q$, $f(p,q)$ is the expected cost of finding the hider. For $(V,\vec{c})$, $P(\vec{c})$ and $Q(\vec{c})$ are the sets of optimal strategies of Players 1 and 2 respectively. Let $b(i) \equiv 1 + c(i)$ for $i = 1,\ldots,n$. $b(i)$ is the sum of the traveling cost and the cost of examining the node i. This quantity is frequently more convenient in analysis rather than $c(i)$.

3. Results.

Theorem 1. Assume $b(i) = k^{i-1}b(1)$ for all $i \in V\backslash\{0\}$ and for $k > 0$. The value of the game is :

$$v(\vec{c}) = \frac{b(1)}{1+k} \sum_{x=1}^{n+1} k^{x-1}.$$

An optimal strategy of the hider is :$p^*(i) = \frac{k^{i-1}}{\sum_{x=1}^{n} k^{x-1}}$ for $i = 1,\ldots,n$. An optimal strategy of the seeker is q^* such that $q^*(\sigma^*) = \frac{1}{1+k}, q^*(r\sigma^*) = \frac{k}{1+k}$ and $q^*(\sigma) = 0$ for $\sigma \neq \sigma^*, r\sigma^*$.

Corollary 2. Assume $c(i) = c > 0$ for all $i \in V\backslash\{0\}$. The value of the game is $\frac{n+1}{2}(1+c)$. An optimal strategy of the hider is $p^* = (\frac{1}{n},\ldots,\frac{1}{n})$. An optimal stratgey of the seeker is q^* such that $q^*(\sigma^*) = q^*(r\sigma^*) = \frac{1}{2}$ and $q^*(\sigma) = 0$ for $\sigma \neq \sigma^*, r\sigma^*$.

Theorem 3. For a game $(V,\vec{c})$, assume $p \in P(\vec{c})$. Then $p(i) > 0$ for all $i \in V\backslash\{0\}$.

Theorem 4. For a game $(V,\vec{c})$, assume $c(i) = c > 0$ for all $i \in V\backslash\{0\}$. If $q \in Q(\vec{c})$, then $q(\sigma) = 0$ for $\sigma \neq \sigma^*, r\sigma^*$.

Theorem 5. For a game $(V,\vec{c})$, suppose $q \in Q(\vec{c})$. If $q(\sigma) > 0$ and $\sigma \neq \sigma^*, r\sigma^*$, then $q(r\sigma) = 0$.

References.

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