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Proximity Theorems of Discrete Convex Functions

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In the area of discrete optimization, nonlinear optimization problems have been investigated as well as linear optimization problems. Submodular (set) functions and separable convex functions are well-known examples of tractable nonlinear functions, in that the submodular function minimization problem can be solved in polynomial time, and separable convex functions have been treated successfully in many different discrete optimization problems.

Recently, certain classes of "discrete convex functions" were proposed: integrally convex functions of Favati and Tardella (1990) and $\{L,M,L_2,M_2\}$-convex functions of Murota (1996,1998). L-convex functions contain the class of submodular set functions. M-convex functions possess structures of matroids and polymatroids. Separable discrete convex functions can be characterized as functions with both L-convexity and M-convexity (in their variants). L_2-convex functions and M_2-convex functions constitute larger classes of discrete convex functions that are relevant to the polymatroid intersection problem, where an L_2-convex function is, by definition, the infimal convolution of two L-convex functions and an M_2-convex function is the sum of two M-convex functions. The M_2-convex function minimization problem is equivalent to the M-convex submodular flow problem which is an extension of the submodular flow problem. The class of integrally convex functions contains all of the above classes.

Those classes C of discrete convex functions f possess the following features in common:

Discreteness: f is defined on an integral lattice $\mathbf{Z}^n$, i.e., $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$, where $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of integers and reals, respectively.

Convex Extendibility: There exists a continuous convex function $\overline{f}$ such that $\overline{f}(x) = f(x)$ for all $x \in \mathbf{Z}^n$.

Optimality Criterion: There exists a neighborhood $N_C(x^*) \subset \mathbf{Z}^n$ with center x^* such that $f(x^*) \leq f(x)$ $(\forall x \in \mathbf{Z}^n)$ if and only if $f(x^*) \leq f(x)$ $(\forall x \in N_C(x^*))$.

Optimality criterion says that global minimality is implied by local minimality defined in terms of the neighborhood $N_C(x^*)$. This is a significant feature inherited from continuous convex functions.

Moreover, L-/M-convex functions have a "proximity property" described as

Proximity Property: Given a positive integer α and a point $x^\alpha \in \mathbf{Z}^n$, there exists a function $d_C(n, \alpha)$ such that

$$\begin{array}{l} f(x^\alpha) \leq f(x) \ (\forall x \in N_C^\alpha(x^\alpha)) \ \Rightarrow \\ \exists x^* \in \arg\min f : \ ||x^* - x^\alpha||_\infty \leq d_C(n, \alpha), \end{array}$$

where

$$N_C^\alpha(x^\alpha) = \{x^\alpha + \alpha(x - x^\alpha) \mid x \in N_C(x^\alpha)\}$$

and $\arg\min f$ denotes the set of all minimizers of f, i.e.,

$$\arg\min f = \{x \in \mathbf{Z}^n \mid f(x) \leq f(y) \ (\forall y \in \mathbf{Z}^n)\}.$$

The proximity property says that a locally minimal solution x^α of a "scaled" function

$$f^\alpha(x) = f(x^\alpha + \alpha x) \qquad (x \in \mathbf{Z}^V)$$

is close to a minimizer x^* of f in terms of $d_C(n, \alpha)$. For L-/M-convex functions,

$$d_C(n, \alpha) = (n-1)(\alpha - 1)$$

is a valid choice ([2] and [3], respectively). The proximity property can be exploited in developing an efficient scaling algorithm for minimizing f. In fact, the L-convex function minimization can be solved in polynomial-time by combining submodular set function minimization algorithms and the proximity property [1] (see [4]). For the M-convex function minimization, polynomial-time scaling algorithms based on the proximity property and its generalization are known [6, 7].

This talk addresses proximity properties of L_2-/M_2-convex functions (see [5] for details). Our main results say:

[L_2 proximity] for an essentially bounded L_2-convex function f and a positive integer α, if $x^\alpha \in \mathrm{dom}\, f$ satisfies

$$f(x^\alpha) \leq f(x^\alpha + \alpha \chi_S)$$

for all $S \subseteq V$, then there exists $x^* \in \arg\min f$ such that

$$\|x^* - x^\alpha\|_\infty \leq 2(n-1)(\alpha-1),$$

[M_2 proximity] for an M_2-convex function f represented as the sum of two M-convex functions f_1 and f_2, and a positive integer α, if $x^\alpha \in \mathrm{dom}\, f$ satisfies

$$\sum_{i=1}^{k}(f_1(x^\alpha - \alpha\chi_{u_i} + \alpha\chi_{w_i}) - f_1(x^\alpha)) + \sum_{i=1}^{k}(f_2(x^\alpha - \alpha\chi_{u_{i+1}} + \alpha\chi_{w_i}) - f_2(x^\alpha)) \geq 0$$

for any ordered subsets $U = \{u_1, \ldots, u_k\}$ and $W = \{w_1, \ldots, w_k\}$ of V with $U \cap W = \emptyset$ where $u_{k+1} = u_1$, then there exists $x^* \in \arg\min f$ such that

$$\|x^* - x^\alpha\|_\infty \leq \frac{n^2}{2}(\alpha - 1).$$

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