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整数計画問題に対するtest setの計算について
Computing test sets for integer programming problems

東京理科大学 ＊伊藤 雅史 ITO Masafumi
東京理科大学 平林 隆一 HIRABAYASHI Ryuichi

1 Introduction

Gröbner bases, introduced by Buchberger in '65 [2], are bases of ideals on the polynomial ring and they have played important roles in commutative algebra and algebraic geometry. In '91, Conti and Traverso proposed an algorithm (the *C-T algorithm*) for integer programmings (IP) using Gröbner bases. Gröbner bases and the C-T algorithm are said to be very powerful tools to analyse IP from the view point of commutative algebra [5]. On the other hand, test sets for IP are introduced by Graver in 1975. In 1995, Thomas probed that Gröbner bases used in C-T algorithm satisfies the conditions of test sets. However, Gröbner bases in C-T algorithm contains a lot of unnecessary elements in general. In 1997, Thomas and Weismantel defined truncated Gröbner bases, in which unnecessary elements are exclude. They also proposed an algorithm to compute them, but it is based on the *Buchberger algorithm* which is known as the extremely time-consuming procedure. In this paper, we propose an another algorithm to compute test sets, based on *the basis conversion techniques.*

2 Test sets for IP

Basic definitions of Gröbner bases are in [1]. We study IP of the form $\mathrm{IP}_{A,c}(b) = \min\{c \cdot x \ : \ Ax = b, x \in \mathbb{N}^n\}$, where $A = [a_{ij}] \in \mathbb{N}^{d \times n}$, $b \in \mathbb{N}^d$, and $c \in \mathbb{R}^n$. We assume $\{x \geq 0 \ : \ Ax = 0\} = \{0\}$ holds.

Definition. The set $T \subset \mathbb{Z}^n \backslash \{0\}$ is called a test set for $\mathrm{IP}_{A,c}(b)$ if it satisfies:
(i) If α is feasible and non-optimal then there exists $u \in T$ such that $\alpha - u$ is feasible and $c \cdot \alpha > c \cdot (\alpha - u)$,
(ii) If β is an optimal solution then $\beta - u$ is infeasible for all $u \in T$.

Now we explain the C-T algorithm for $\mathrm{IP}_{A,c}(b)$. Consider the polynomial ring $k[t, y] := k[t_1, \ldots, t_d, y_1, \ldots, y_n]$. $\succ_{(t>y)}$ is a term order satisfying that all terms with t_i are larger than that contain only y_j. Furthermore, $\succ_c$ is a term order satisfying $c \cdot \alpha > c \cdot \beta \Rightarrow y^\alpha \succ_c y^\beta$. In the C-T algorithm, we have to compute a Gröbner basis G of an ideal $\langle F \rangle \subset k[t, y]$ *w.r.t.* a term order with properties of $\succ_{(t>y)}$ and $\succ_c$. An optimal solution can be obtained from the remainder of $t_1^{b_1} \ldots t_d^{b_d}$ divided by G.

It is easy to see that the Gröbner basis G satisfies the conditions of test sets. The crucial fact is that G is a subset of $\{x^u - x^v \ : \ Au = Av\}$. Thomas and Weismantel proved that the set $bG := \{x^u - x^v \in G \ : \ b - u \in C_{\mathbb{N}}(A)\}$ also satisfies the condition of test sets, where $C_{\mathbb{N}}(A) := \{Au \ : \ u \in \mathbb{N}^n\}$. This bG is called *a b-Gröbner basis* for $\mathrm{IP}_{A,c}(b)$ [5]. Our objective is to construct an algorithm to compute bG.

3 Our Algorithm

It consists of *PHASE 1* and *PHASE 2*.

In general, the number of variables is desired to be small for Gröbner bases computations. Hence the goal of PHASE 1 is to eliminate the variables $t_1, \ldots, t_d$ from $\langle F \rangle$, that are unnecessary in bG. It can be done by using an elimination property of Gröbner bases of $\langle F \rangle$ *w.r.t.* $\succ_{(t>y)}$. We focus our attention to the fact that F is already a Gröbner basis *w.r.t.* an undesired order $\succ_{\times}$. The change of ordering of Gröbner bases from $\succ_{\times}$ to $\succ_{(t>y)}$ can be done by the *Gröbner Walk* [3], but it is not fast enough according to our computational result. Our procedure, called as the *b-FGLM*, computes $\tilde{F} \in k[t, y]$, which is a subset of a Gröbner basis of $\langle F \rangle$ *w.r.t.* $\succ_{(t>y)}$. $\tilde{F}$ keeps an elimination property of Gröbner bases, and has an additional property that $B := \tilde{F} \cap k[y]$ is a basis of the linear space on k spanned by bG. The procedure is based on the *FGLM* algorithm [4], which is a Gröbner bases conversion algorithm for 0-dimensional ideals, and said to be very fast when handling binomial ideals. In our case, $\langle F \rangle$ is a binomial ideal, but unfortunately, not 0-dimensional. The difficulty is that FGLM does not terminate when ideals are not 0-dimensional. We overcome this difficulty by making use of the multivariate grading on $k[t, y]$ defined in [5], and ensure its termination by the finiteness of feasible solutions of $\mathrm{IP}_{A,c}(b)$.

In PHASE 2, B is converted into bG by our procedure called *b-Gröbner Walk*. The main strategy is similar to the original. We define the *b-Gröbner cone* for each term order in the same way to the definition of the Gröbner cone. Moreover, the *departure* w_0 is set to $(1, 0, \ldots, 0)$. By tracing the line segment from w_0 to c, we can find a boundary w_1 of the current b-Gröbner cone. Then we transform B into a basis *w.r.t.* the term order of the next Gröbner cone by the local conversion procedure, and trace again from w_1 to c. Repeat this operation until the line segment w_i to c is in the same b-Gröbner cone. In the local conversion, we have to apply the *b-Buchberger algorithm* introduced in [5]. It is based on the *Buchberger algorithm* [1] by adding the *exclusion test* not to generate unnecessary elements for $\mathrm{IP}_{A,c}(b)$. Generally, the exclusion test is not practical because the exclusion test requires to solve IPs of the form $\mathrm{IP}_{A,b}(\beta)$. But in our algorithm, this test can be done quite easily by dividing $t_1^{\beta_1} \ldots t_d^{\beta_d}$ by $\tilde{F}$.

4 Conclusion

We combined the theory of b-Gröbner bases with the bases conversion technique. Our algorithm will help us to analyze larger size of IPs and related problems algebraically.

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