Markov renewal functions in the $M/G/1$ type queues

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1. Introduction

We consider Markov renewal equations and their solutions arising in the stationary distribution of the $M/GI/1$ type queue (e.g. see [3]). It is shown that the transition kernels of these renewal equations can be expressed by ladder epochs of a Markov additive process that describes the system queue length when the system is not empty. We also consider how variants of the Markov renewal function, e.g., the one in Takine [5], arise. Usually, matrix analysis is used for studying the $M/G/1$ type queue. Unlike this, we mainly use stochastic arguments, which not only simplify proofs but also reveal new aspects.

2. Ramaswami's identity

Let S be a finite set. Let $A(n)$ and $B(n)$, $n = 0, 1, \ldots$, be $S \times S$ nonnegative matrices such that $\sum_{n=0}^{\infty} A(n)\boldsymbol{e} = \boldsymbol{e}$ and $\sum_{n=0}^{\infty} B(n)\boldsymbol{e} = \boldsymbol{e}$, where $\boldsymbol{e}$ is S-column vector all of whose entries are unit. Let $\mathbb{Z}_+ = \{0, 1, \ldots\}$ and $S_1 = \mathbb{Z}_+ \times S$. Define the $S_1 \times S_1$ transition probability matrix P as

$$P = \begin{pmatrix} B(0) & B(1) & B(2) & B(3) & \cdots \\ A(0) & A(1) & A(2) & A(3) & \cdots \\ 0 & A(0) & A(1) & A(2) & \cdots \\ . & . & . & . & \cdots \\ . & . & . & . & \cdots \end{pmatrix}.$$

The Markov chain with this P is referred to as the $M/G/1$ *type*. Let $\boldsymbol{x}(n)$, $n = 0, 1, \ldots$, be nonnegative S-row vectors. Then, $\boldsymbol{x} \equiv \{\boldsymbol{x}(n); n \geq 0\}$ is said to be the stationary measure of P if $\boldsymbol{x}P = \boldsymbol{x}$, equivalently,

$$\boldsymbol{x}(n) = \boldsymbol{x}(0)B(0) + \sum_{\ell=1}^{n+1} \boldsymbol{x}(\ell)A(n+1-\ell), \qquad (2.1)$$

and, in particular, said to be the stationary distribution if $\sum_{n=0}^{\infty} \boldsymbol{x}(n)\boldsymbol{e} = 1$.

We next introduce the Markov chain obtained from P removing boundary states $\{0\} \times S$ and extending the state space from S_1 to $\mathbb{Z} \times S$, where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Denote this Markov chain by $\{(Y_n, X_n)\}$. From the structure of P, Y_n is skip free in the downward.

Let $\tau_n = \inf\{\ell \geq 1; Y_\ell = -n\}$. Since $\{Y_n\}$ is skip free, $\{X_{\tau_n}\}$ is also a Markov chain. Denote the $S \times S$ transition matrix of this Markov chain by G. Then, we have $G = \sum_{n=0}^{\infty} A(n)G^n$, which is called the *fundamental matrix*. Let $\Phi_A(n) = \sum_{\ell=n}^{\infty} A(\ell)G^{\ell-n}$ and $\Phi_B(n) = \sum_{\ell=n}^{\infty} B(\ell)G^{\ell-n}$. Ramaswami [4] shows that, if $\boldsymbol{x} \equiv \{\boldsymbol{x}(n)\}$ is the stationary distribution, then it satisfies, for $n \geq 0$,

$$\boldsymbol{x}(n) = \boldsymbol{x}(0)\Phi_B(n) + \sum_{\ell=1}^{n} \boldsymbol{x}(\ell)\Phi_A(n+1-\ell), \quad (2.2)$$

This equation is rearranged to

$$\boldsymbol{x}(n) = \left[\boldsymbol{x}(0)\Phi_B(n) + \sum_{\ell=1}^{n-1} \boldsymbol{x}(\ell)\Phi_A(n+1-\ell)\right] \times (I - \Phi_A(1))^{-1}, \quad n \geq 1. \quad (2.3)$$

We here give a new proof for (2.2) without using a censored process as usual.

Lemma 2.1 If (2.2) holds, then $\{\boldsymbol{x}(n)\}$ is the stationary vector of P.

3. Markov renewal equation

Let $\Psi_A(n) = \Phi_A(n+1)$, then (2.2) becomes

$$\boldsymbol{x}(n) = \boldsymbol{x}(0)(\Phi_B(n) - \Phi_A(n+1)) + \sum_{\ell=0}^{n} \boldsymbol{x}(\ell)\Psi_A(n-\ell) \quad n \geq 0. \quad (3.1)$$

This is a Markov renewal equation if $\sum_{n=0}^{\infty} \Psi_A(n)\boldsymbol{e} \leq \boldsymbol{e}$ or $\sum_{n=0}^{\infty} \Psi_A^{\mathrm{T}}(n)\boldsymbol{e} \leq \boldsymbol{e}$. Unfortunately, this may not be true. We convert this equation to the Markov renewal equation using a dual process, defined below.

Since $\{X_n\}$ is a Markov chain with transition rate matrix $A \equiv \sum_{n=0}^{\infty} A(n)$, which is assumed to be irreducible, X_n has the unique stationary distribution, denoted by S-row vector $\boldsymbol{\pi}$. Assume that X_0 is subject to the distribution $\boldsymbol{\pi}$. Then, $\{X_n; n \geq 0\}$ is a stationary process, so we can extend this process on the whole time axis. Since $\{Y_n\}$ is defined on the $\{X_n\}$, we can extend it similarly. We then define the dual process $\{(\tilde{Y}_n, \tilde{X}_n)\}$ by $\tilde{X}_n = X_{-n}$ and $\tilde{Y}_n = -Y_{-n}$. It is easy to see that $\{\tilde{X}_n\}$ is the Markov chain with transition rate matrix: $\tilde{A} = \Delta_{\boldsymbol{\pi}}^{-1} A^{\mathrm{T}} \Delta_{\boldsymbol{\pi}}$, where $\Delta_{\boldsymbol{a}}$ is the diagonal matrix whose i-th entry is the i-th component of a vector $\boldsymbol{a}$. Clearly, $\{\tilde{Y}_n\}$ is a Markov additive process with background process $\{\tilde{X}_n\}$.

Define $\tilde{\tau}^+ = \inf_{n\geq 1}\{n | \tilde{Y}_n \geq 0\}$. That is, $\tilde{\tau}^+$ is the weak ladder epoch of $\{\tilde{Y}_n\}$. The following result is a key observation for our arguments.

Lemma 3.1 Under any drift condition, we have

$$P(\tilde{Y}_{\tilde{\tau}^+} = \ell, \tilde{X}_{\tilde{\tau}^+} = j | \tilde{X}_0 = i) = \frac{\pi_j}{\pi_i} [\Phi_A(\ell+1)]_{ji} . \quad (3.2)$$

We now convert (3.1) to a Markov renewal equation. Let $\tilde{\boldsymbol{x}}(n) = \Delta_{\boldsymbol{\pi}}^{-1} \boldsymbol{x}(n)^{\mathrm{T}}$, $\tilde{\Phi}_C(n) = \Delta_{\boldsymbol{\pi}}^{-1} \Phi_C(n+1)^{\mathrm{T}} \Delta_{\boldsymbol{\pi}}$ for $C = A, B$ and $\tilde{\Psi}_A(n) = \tilde{\Phi}_A(n+1)$. Then, we have the following result from (3.3) by taking transpose.

Theorem 3.1 If $\beta_A \equiv \sum_{n=1}^{\infty} n A(n) \boldsymbol{e} \leq 1$, then

$$\tilde{\boldsymbol{x}}(n) = (\tilde{\Phi}_B(n) - \tilde{\Phi}_A(n+1))\tilde{\boldsymbol{x}}(0) + \sum_{\ell=0}^{n} \tilde{\Psi}_A(n-\ell)\tilde{\boldsymbol{x}}(\ell), \quad n \geq 0, \quad (3.3)$$

is the Markov renewal equation, and has the solution $\{\tilde{\boldsymbol{x}}(n)\}$. Furthermore, $\{\boldsymbol{x}(n)\} = \{\tilde{\boldsymbol{x}}(n)^{\mathrm{T}} \Delta_{\boldsymbol{\pi}}\}$ is the stationary measure of P, and the Markov renewal kernel $\{\tilde{\Psi}_A(n); n \geq 0\}$ is proper only if $\beta_A = 1$, so $\{\boldsymbol{x}(n)\}$ is a probability distribution only if $\beta_A < 1$.

Remark 3.1 A standard setting in the $M/G/1$ type queue assumes that $B(0) = A(0) + A(1)$ and $B(n) = A(n+1)$ for $n \geq 1$. In this case, we have $\tilde{\Phi}_B(n) - \tilde{\Phi}_A(n+1) = A(0)1(n=0)$.

Denote the Markov renewal measure for the kernel $\{\tilde{\Psi}_A(n); n \geq 0\}$ by $\tilde{U}(n) = \sum_{\ell=0}^{\infty} \tilde{\Psi}_A^{(*\ell)}(n)$,where matrix convolution $A * B(n)$ is defined as $[A * B(n)]_{ij} = \sum_{\ell=0}^{n} \sum_k [A(\ell)]_{ik} [B(n-\ell)]_{kj}$, and $\tilde{\Psi}_A^{(*\ell)}(n) = I$ for $\ell = 0$. Hence, if $\beta_A < 1$, the stationary distribution $\{\boldsymbol{x}(n)\}$ is obtained as, using $U(n) = \sum_{\ell=0}^{\infty} \Psi_A^{(*\ell)}(n)$,

$$\boldsymbol{x}(n) = \boldsymbol{x}(0)(\Phi_B - \Psi_A) * U)(n), \qquad n \geq 0. \quad (3.4)$$

4. Alternative formulations

There are several variants of the renewal equations and functions. In this section, we discuss how they arise. Generally speaking, those variants come from different choices of the Markov renewal kernel and the initial term of the sequence $\{\tilde{\boldsymbol{x}}(n)\}$.

(Modifying the initial term) Let the sequence in (3.3) starts with $n = 1$, then, for $n \geq 1$,

$$\tilde{\boldsymbol{x}}(n) = \tilde{\Phi}_B(n)\tilde{\boldsymbol{x}}(0) + \sum_{\ell=1}^{n} \tilde{\Psi}_A(n-\ell)\tilde{\boldsymbol{x}}(\ell). \quad (4.1)$$

Thus, we have $\boldsymbol{x}(n) = \boldsymbol{x}(0)(\Psi_B * U)(n-1)$ for $n \geq 1$, where $\Psi_B(n) = \Phi_B(n+1)$. Note that $U(0)$ is not null but $U(0) = \sum_{\ell=0}^{\infty} \Psi_A^{\ell}(0) = (I - \Phi_A(1))^{-1}$.

(Modifying the Markov renewal kernel) We next consider to use the Markov renewal kernel corresponding with (2.3). Then, for $n \geq 1$,

$$\tilde{\boldsymbol{x}}(n) = (I - \tilde{\Phi}_A(1))^{-1} \tilde{\Phi}_B(n)\tilde{\boldsymbol{x}}(0) + \sum_{\ell=1}^{n-1} (I - \tilde{\Phi}_A(1))^{-1} \tilde{\Psi}_A(n-\ell)\tilde{\boldsymbol{x}}(\ell), \quad (4.2)$$

This is the renewal equation, which is equivalent to the one that is obtained by Takine [5]. In this case, the Markov renewal kernel is given by $\tilde{\Gamma}_A(n) = (I - \tilde{\Phi}_A(1))^{-1} \tilde{\Psi}_A(n)$. This is a right kernel since Lemma 3.1 implies $\sum_{n=1}^{\infty} \tilde{\Phi}_A(n) \boldsymbol{e} \leq \boldsymbol{e}$. Denote the Markov renewal measure for the kernel $\{\tilde{\Gamma}_A(n)\}$ by $\tilde{V}(n)$. Then, we have $\boldsymbol{x}(n) = \boldsymbol{x}(0)(\Gamma_B * V)(n-1)$ for $n \geq 1$, where $\Gamma_B(n) = \Phi_B(n)(I - \Phi_A(1))^{-1}$.

The advantage of this kernel is that it easily handles the tail probabilities defined by $\overline{\boldsymbol{x}}(n) = \sum_{\ell=n}^{\infty} \boldsymbol{x}(\ell)$. Namely, we have $\overline{\boldsymbol{x}}(n) = \boldsymbol{x}(0)(\overline{\Gamma}_B * V)(n-1)$ for $n \geq 1$, where $\overline{\Gamma}_B(n) = \sum_{\ell=n}^{\infty} \Gamma_B(\ell)$.

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