2002年日本オペレーションズ・リサーチ学会
秋季研究発表会

Portfolio Optimization under Short Sale Opportunity

01102370 中央大学 今野浩 KONNO Hiroshi
02103820 中央大学 *越塚知幸 KOSHIZUKA Tomoyuki
02702020 中央大学 山本零 YAMAMOTO Rei

1. Introduction

We consider a portfolio optimization problem under short sale opportunity. When we sell assets short we must pay deposit and commission fee to the third party who lends the assets, and the cash obtained by the short sale is held at the party. We have to use up to all amount of fund at the time of portfolio construction. In this case the investable set is a nonconvex set, so that the mean-variance model becomes a nonconvex minimization problem. This kind of problem cannot be solved by standard nonlinear programming methodology. So we propose a branch and bound algorithm exploiting the special structure of this problem. It is demonstrated that this algorithm can solve virtually all test problems in a very efficient manner.

2.Formulation

Let there be n assets $S_j, j = 1, 2, \cdots, n$ in the market and let R_j be the random variable representing the rate of return of S_j. Also, let $x_j, j = 1, 2, \cdots, n$ be the proportion of fund (either positive or negative) to be allocated to S_j. The total cash outflow is (γ is a positive constant)

$$\sum_{j=1}^{n}\{|x_j|_+ + \gamma|x_j|_-\} = 1 \qquad (1)$$

where

$$|x_j|_+ = \begin{cases} x_j & \text{if } x_j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
$$|x_j|_- = \begin{cases} 0 & \text{if } x_j \geq 0 \\ -x_j & \text{otherwise} \end{cases}.$$

Let us introduce u_j and v_j such that

$$u_j - v_j = x_j,\ u_j \geq 0,\ v_j \geq 0,\ u_j v_j = 0, \quad j = 1, 2, \cdots, n.$$

Then $|x_j|_+ = u_j$, $|x_j|_- = v_j$. We employ the historical data to represent the return structure of the asset. The mean-variance model can be represented as follows:

$$\begin{array}{ll}
\text{maximize} & f(u,v,y) = \sum_{j=1}^{n} r_j(u_j - v_j) \\
 & \quad -\lambda \sum_{t=1}^{T} y_t^2/T - \sum_{j=1}^{n}(c_j^+ u_j + c_j^- v_j) \\
\text{subject to} & y_t - \sum_{j=1}^{n}\beta_{jt}(u_j - v_j) = 0, \\
 & \quad t = 1,2,\cdots,T \\
 & \sum_{j=1}^{n} u_j + \gamma \sum_{j=1}^{n} v_j = 1 \\
 & \sum_{j=1}^{n} a_{ij}(u_j - v_j) \geq b_i, \\
 & \quad i = 1,2,\cdots,m \\
 & 0 \leq u_j \leq \alpha, \quad 0 \leq v_j \leq \alpha' \\
 & \quad u_j v_j = 0, \quad j = 1,2,\cdots,n
\end{array}$$

where $r_{jt}, t = 1,2,\cdots,T$ are the rate of return of the j-th asset during the past t-th period. Also, $\beta_{jt} = r_{jt} - r_j$.

To construct a practical algorithm, we relax the equality constraint (1) as follows:

$$1 - \delta \leq \sum_{j=1}^{n} u_j + \gamma \sum_{j=1}^{n} v_j \leq 1 \qquad (2)$$

for some positive constant $\delta > 0$.

3. A Branch and Bound Algorithm

The first and natural step for solving this nonconvex problem is to relax the complementarity condition $u_j v_j = 0, j = 1,2,\cdots,n$ and solve the resulting quadratic programming problem. Let (u^*, v^*, y^*) be an optimal solution of the quadratic programming problem. If $u_j^* v_j^* = 0$ $,j = 1,2,\cdots,n$, then it is obviously an optimal solution of the original problem. If there exist any $j's$ such that violate complementarity condition, we use a branch and bound method. We consider the following linear system

$$\left|\begin{array}{ll}\sum_{j=1}^{n}\beta_{jt}u_j-\sum_{j=1}^{n}\beta_{jt}v_j=y_t^*, & t=1,2,\cdots,T\\ \sum_{j=1}^{n}a_{ij}u_j-\sum_{j=1}^{n}a_{ij}v_j\geq b_i, & i=1,2,\cdots m\\ 1-\delta\leq\sum_{j=1}^{n}u_j+\gamma\sum_{j=1}^{n}v_j\leq 1 & \\ 0\leq u_j\leq\alpha,\ 0\leq v_j\leq\alpha', & j=1,2,\cdots,n.\end{array}\right.$$

Obviously this problem has a feasible solution, so that it has a basic feasible solution $(\hat{u},\hat{v})$.

Let $\hat{u_k}>0$, $\hat{v_k}>0$. If $\hat{u_k}\hat{v_k}>\epsilon$,then let us redefine

$$\begin{aligned}\hat{u_k}&=\begin{cases}\hat{u_k}-\hat{v_k} & \text{if } \hat{u_k}\geq\hat{v_k}\\ 0 & \text{otherwise,}\end{cases}\\ \hat{v_k}&=\begin{cases}0 & \text{if } \hat{u_k}>\hat{v_k}\\ \hat{v_k}-\hat{u_k} & \text{otherwise.}\end{cases}\end{aligned}\tag{3}$$

Therefore if the new solution satisfy the condition(2) then we are done. Otherwise we split the problem into two subproblems by imposing the condition $u_k=0$ or $v_k=0$.

Let us define

$$(P_l)\left|\begin{array}{ll}\text{maximize} & f(u,v,y)\\ \text{subject to} & y_t-\sum_{j=1}^{n}\beta_{jt}(u_j-v_j)=0,\\ & \quad t=1,2,\cdots,T\\ & 1-\delta\leq\sum_{j=1}^{n}(u_j+\gamma v_j)\leq 1\\ & 0\leq u_j\leq\alpha,\quad 0\leq v_j\leq\alpha',\\ & \quad j=1,2,\cdots,n\\ & u_j=0,\ j\in U_l,\quad v_j=0,\ j\in V_l\end{array}\right.$$

$$Q_l(y^*)\left|\begin{array}{ll}\sum_{j=1}^{T}\beta_{jt}(u_j-v_j)=y_t^*, & t=1,2,\cdots,T\\ 1-\delta\leq\sum_{j=1}^{n}(u_j+\gamma v_j)\leq 1 & \\ 0\leq u_j\leq\alpha,\quad 0\leq v_j\leq\alpha', & j=1,2,\cdots,n\\ u_j=0,\ j\in U_l, & \\ v_j=0,\ j\in V_l & \end{array}\right.$$

Algorithm (Branch and Bound Algorithm)

0° $\epsilon>0$; $U_0=V_0=\phi$; $l:=0$, $f^*:=-\infty$, $\mathscr{P}=\{P_0\}$

1° If $\mathscr{P}=\phi$ then go to 7°.

2° Choose one subproblem $P_l\in\mathscr{P}$ and let $\mathscr{P}\setminus\{P_l\}$.

3° Solve (P_l). If (P_l) is infeasible, then go back to 1°. Otherwise let (u^l, v^l, y^l) be its optimal solution and let f_l be its optimal value.

4° Culculate a basic feasible solution $(\hat{u_j}^l, \hat{v_j}^l)$ of the linear system $Q_l(y^l)$.

5° If the complementarity conditions are satisfied, then let f_l be its optimal value. Otherwise generate a new solution by (3). If the solution satisfies (2), then let f_l be the associated optimal value. If $f_l>f^*$, then $f^*:=f_l$ and remove all subproblem P_h such that $f_h\leq f^*$.

6° Let

$$\begin{aligned}U_{L+1}&=U_l\cup\{s\,|\max\{\hat{v}_s^l\hat{u}_s^l>\varepsilon\}\}\\ V_{L+1}&=V_l\cup\{s\,|\max\{\hat{v}_s^l\hat{u}_s^l>\varepsilon\}\}\end{aligned}$$

and generate subproblems P_{L+1} and P_{L+2}. Let $\mathscr{P}=\mathscr{P}\cup\{P_{L+1},P_{L+2}\}$, and return to 1°.

7° Stop.

4.Computational Result and Conclusions

We tested the Algorithm for this problem with and without upper bound on eash variable, using the monthly data of the Tokyo Stock Exchange.We show computational results in presentation.

The problem without upper bound (Class 1) is easier since the solution generated at the first stage contains at most one pair of variable violating the complementarity condition. All test problems were solved without using branch and bound procedure.

While the problem with upper bound (Class 2) is more complicated. However, we show that only one out of six test problems required branch and bound procedure.

We demonstrated that a mean-variance model under short sale opportunity can be solved very fast. Though nonconvex, it is usually not more difficult than solving standard mean-variance model without short sale opportunity. Also even when we need to apply branch and bound procedure, the number of iterations is not excessive since the number of variables violating the complementarity condition is usually very small. We are currently planning extensive experiments to compare the relative advantage of the MV model with and without short sale opportunity using the factor model.

Reference. 今野浩「理財工学Ⅰ，Ⅱ」日科技連出版,1995,1998