2002年日本オペレーションズ・リサーチ学会
秋季研究発表会

# Monte Carlo Analysis of Convertible Bonds with Reset Clauses

01105381 北海道大学 木村 俊一 KIMURA Toshikazu
入会申請中 北海道大学 *篠原 敏夫 SHINOHARA Toshio

## 1 Introduction

Convertible bonds or more generally equity-linked securities have greatly evolved in the past decade. Convertible bonds are hybrid securities issued by a firm where the holder has the right to convert the bond into the common stocks of the firm according to pre-specified conditions. An essential feature of ordinary convertible bonds is that they may be converted at any time until a pre-specified maturity date into stocks at a pre-specified ratio, *i.e.*, a fixed *conversion ratio*. In recent years, however, various convertible bonds have been issued with additional conversion provisions. Among others, some Japanese bank convertibles have a *reset* clause whereby the conversion ratio is adjusted upwards, or equivalently, the conversion price adjusted downwards if the underlying stock price does not exceed pre-specified trigger prices.

The price of any convertible bonds ($V_{\rm CB}$) can be *approximately* viewed as a sum of values of an otherwise identical non-convertible bond ($V_{\rm SB}$) plus an embedded option to convert the bond into the underlying stock ($V_{\rm CO}$). In general, these two components interact with each other and so prove to difficult to separate. However, in some situations, the embedded option can be separated and easily valued. A separable case is, for example, that the convertible bond is non-callable and non-convertible until maturity. As a basic framework for pricing, we use the bond plus option valuation whether or not the underlying stock has credit risk of the issuer. We principally focus on the price of conversion option, which is essential in analyzing the price of convertible bonds under the constant interest-rate assumption.

## 2 Exact Analysis of Credit-Riskless Convertible Bonds

For a class of non-callable convertible bonds with the reset clause, we first consider a special case such that the issuer has no credit risk. Under the assumptions of no dividends on the underlying stock and the flat term structure of the risk-free interest rate, no conversion occurs prior to maturity. Assume that the capital market is well-defined and follows the efficient market hypothesis. Let $S_t$ denote the underlying stock price at time $t$ and assume a geometric Brownian motion model

$$dS_t = S_t(rdt + \sigma dW_t), \qquad 0 \le t \le T. \qquad (1)$$

The interest rate $r$, the volatility $\sigma$ and the maturity $T$ of the convertible bond are assumed to be positive constants. The process $W \equiv \{W_t; 0 \le t \le T\}$ is the standard Brownian motion process under a probability measure $\mathbb{P}$ which is *risk-neutral*, *i.e.*, is chosen so that the stock has mean rate of return $r$.

Let $K$ $(> 0)$ be the *original* conversion price, and let $\tau$ $(\in (0,T))$ be the reset time. Then, the *actual* conversion price of the reset convertible is changed to $aS_\tau$ if $S_\tau < K/a$ for a pre-specified constant $a \ge 1$, or it remains $K$ otherwise.

Assume that the convertible bond receives a coupon of amount $C$ $(> 0)$ at time $T_i$ $(i = 1, \ldots, n)$ where $0 < T_1 < T_2 < \cdots < T_n \le T$. For the straight bond value $V_{\rm SB}$, we immediately have $V_{\rm SB} = C\sum_{i=1}^{n} e^{-rT_i} + Be^{-rT}$. On the other hand, for the conversion option price $V_{\rm CO}$, we have

**Theorem 1** *Let $V_{\rm CO}$ be the conversion option value of the credit-riskless, non-callable, convertible bond with the reset clause at time $t = 0$. Then,*

$$\begin{aligned} V_{\rm CO} &= S_0\Big(\Phi(d_1^+) - ae^{-r(T-\tau)}\Phi(d_1^-)\Big)\Phi(-d_0^+) \\ &\quad + \int_{\ln(K/aS_0)}^{\infty}\Big(S_0e^{-r\tau}\Phi(d_2^+(y))e^y \\ &\quad - Ke^{-rT}\Phi(d_2^-(y))\Big)\psi(y)dy, \end{aligned} \qquad (2)$$

*where $\Phi(\cdot)$ is the cdf of the standard normal distribution,* i.e., *for $x \in \mathbb{R}$*

$$\Phi(x) = \int_{-\infty}^{x}\phi(y)dy \quad \textit{with} \quad \phi(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}},$$

$$\psi(y) = \frac{1}{\sigma\sqrt{\tau}}\phi\left(\frac{y-(r-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right),$$

*and the parameters $d_0^+$, $d_1^\pm$ and $d_2^\pm(y)$ are defined by*

$$d_0^+ = \frac{\ln(aS_0/K) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},$$

$$d_1^\pm = \frac{-\ln a + (r\pm\frac{1}{2}\sigma^2)(T-\tau)}{\sigma\sqrt{T-\tau}},$$

$$d_2^\pm(y) = \frac{y + \ln(S_0/K) + (r\pm\frac{1}{2}\sigma^2)(T-\tau)}{\sigma\sqrt{T-\tau}}.$$

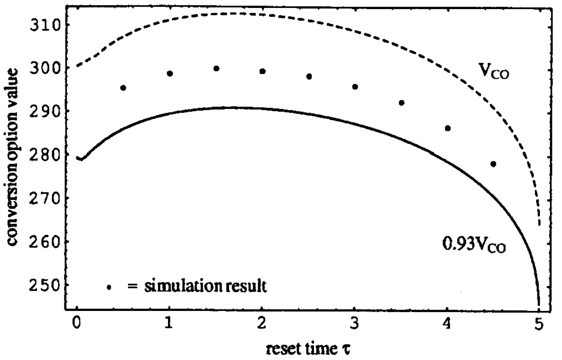


Figure 1: Conversion Option Values of Credit-Risky Reset Convertible Bonds

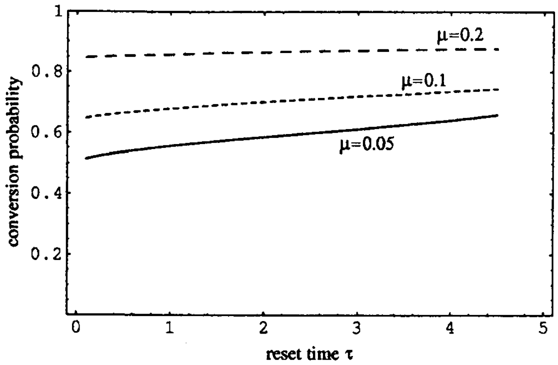


Figure 2: Conversion Probabilities of Credit-Risky Reset Convertible Bonds

## 3 Simulation Analysis of Credit-Risky Convertible Bonds

We use Monte Carlo simulation to analyze features of the credit-risky and non-callable convertible bond with the reset clause. The first step in Monte Carlo simulation is to generate sample paths of the process $S \equiv \{S_t;\, 0 \le t \le T\}$. Let $N$ be the total number of sample paths and let $M$ be the number of time steps in the following discrete-time version of (1):

$$S_{ij} = S_{i,j-1}(1 + r\Delta t + \sigma\sqrt{\Delta t}\xi_{ij}), \qquad (3)$$

for $j = 1, \ldots, M$, $i = 1, \ldots, N$, where $\Delta t = T/M$ and $S_{ij}$ is the simulated stock price at time $t_j \equiv j\Delta t$ ($j = 0, \ldots, M$) in the $i$-th sample path starting from $S_{i0} = S_0$ ($i = 1, \ldots, N$). The variables $\{\xi_{ij}\}$ are iid standard normal random numbers.

As a credit-risk dynamics, assume that defaults may occur depending on the stock price at that time. More specifically, assume that the issuer defaults during the time interval $[t_{j-1}, t_j)$ with probability $\lambda(S_{i,j-1})\Delta t$ given that it survives until time $t_{j-1} < T$, where $\lambda(S_t) \ge 0$ is the instantaneous default rate. For the case that $\lambda(\cdot)$ is independent of $S$, this assumption clearly implies that the time of default, say $D$, is exponentially distributed with parameter $\lambda$, *i.e.*, $\mathbb{P}\{D > t\} = e^{-\lambda t}$ for $t \ge 0$.

To compute the conversion option value, we adopt the Grant-Vora-Week (GVW) method [1].

Figure 1 respectively illustrates simulation results for the conversion option value embedded in a risky convertible bond with the reset clause, where $a = 1$, $T = 5$ and $K = 1100$. Assume $S_0 = 1000$ and $\sigma = 0.3$ for the underlying stock, and $r = 0.02$ for the risk-free interest rate. We used the constant default rate satisfying $\mathbb{P}\{D > T\} = e^{-\lambda T} = 0.93$, *i.e.*, $\lambda = -\ln(0.93)/5 \approx 0.0145$. To compute the locus of critical prices in the modified GVW method, we used $M = 50$ and $N = 500{,}000$, together with a standard antithetic variance reduction technique of coupling a pair of sample paths.

Figure 1 also shows two curves indicating $V_{\rm CO}$ and $\mathbb{P}\{D > T\}V_{\rm CO}$ for the credit-riskless convertible bond with the same parameters.

We see from Figure 1 that the simulation results are located between these two curves, which certainly justifies the theoretical result that the proportional impact of default risk on the price of an American option is less than that for a similar European option. As in the riskless cases, the bond-holder can expect high returns when the reset time is in the former half of the trading interval.

Figure 2 illustrates the conversion probabilities of a risky convertible bond with the reset clause as a function of the reset time $\tau$, by smoothing simulation results. Taking account for the risk premium of the stock, we assumed instead of (1) that the process $S$ is governed by the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma d\hat{W}_t), \qquad 0 \le t \le T, \qquad (4)$$

where $\mu$ ($> r$) is the return rate of the underlying stock and $\hat{W} \equiv \{\hat{W}_t;\, 0 \le t \le T\}$ is a $\hat{\mathbb{P}}$-Brownian motion for a probability measure $\hat{\mathbb{P}}$ in the real world. From Figure 2, we see that the conversion probability is a non-decreasing function of $\tau$, *i.e.*, the later the reset time, the higher the conversion probability. This immediately means that the best reset time for the issuer is $\tau = T$. Combining this fact with the preference of the holder, we can conclude that there exists in the latter half of the trading interval an optimal reset time for both investors and the issuer.

## References


[1] Grant, D., Vora, G. and Week, D., Simulation and the early-exercise option problem, *Journal of Financial Engineering*, **5**, 211–227 (1996).