2002年日本オペレーションズ・リサーチ学会
秋季研究発表会

On the term structure of lending interest rates when a fraction of collateral is recovered upon default

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1 Introduction

This article provides an arbitrage-free model to evaluate the term structure of lending interest rates when a fraction of collateral is recovered upon default of the borrower. Unlike the previous literature, we assume that the value of the collateral asset fluctuates over time with certain correlation to the risk-free interest rate as well as the default hazard rate of the borrower. It is shown that the bank loan is a sum of holding a coupon-bearing bond and selling a put option, both being callable upon default. A Gaussian model is considered, as a special case, to derive an analytic expression of the appropriate lending interest rates, and some numerical example is given to demonstrate that bank loans exhibit different properties from corporate bonds.

2 The Model Framework

Throughout, we fix the probability space $(\Omega, \mathcal{F}, P)$ and denote the expectation operator by E. The probability measure P is the *risk-neutral measure*, since we are interested in the pricing of financial instruments. The canonical filtration generated by the underlying stochastic structure is denoted by $\{\mathcal{F}_t\}$.

Suppose that the current time is 0 and the bank lends F dollars to a corporate firm with maturity $T > 0$. In compensation of the loan, the firm repays αF dollars to the bank at time epochs t_j, where $0 = t_0 < t_1 < t_2 < \cdots < t_m = T$, and the principal F at maturity T in the case of no default before the maturity. If default occurs during $(t_k, t_{k+1}]$, then the payment terminates at time t_k.

Let τ denote the default epoch of the firm, and let $L(t)$ be the time t value of the collateral asset. If the firm defaults before the maturity T, the recovery of the bank is given by $\min\{\beta L(\tau), F\}$ for some β, where $0 \le \beta \le 1$.

Let $r(t)$ denote the default-free spot rate at time t, and denote the *cumulative* spot rate by $R(t) = \int_0^t r(u)\mathrm{d}u$. The time t money-market account is then given by $B(t) = \mathrm{e}^{R(t)}$, $t \ge 0$. According to the risk-neutral method, the present value of the total payments of the firm is obtained, after some algebra, as

$$T_R = E\Big[\alpha \sum_{j=1}^{m} \frac{F}{B(t_j)} 1_{\{\tau > t_j\}} + \frac{F}{B(T)} 1_{\{\tau > T\}} + \frac{\min\{\beta L(\tau), F\}}{B(\tau)} 1_{\{\tau \le T\}}\Big], \quad (1)$$

where 1_A denotes the indicator function. Since

$$\min\{x, y\} = y - \max\{y - x, 0\},$$

we obtain from (1) that

$$T_R = E\Big[\alpha \sum_{j=1}^{m} \frac{F}{B(t_j)} 1_{\{\tau > t_j\}} + \frac{F}{B(T \wedge \tau)}\Big] - E\Big[\frac{\max\{F - \beta L(\tau), 0\}}{B(\tau)} 1_{\{\tau \le T\}}\Big], \quad (2)$$

where $x \wedge y = \min\{x, y\}$. We thus have the following.

Proposition 1 *A bank loan with a collateral asset is equal to a sum of holding a coupon-bearing bond and selling a put option written on the collateral asset, both being callable at default epoch. The lending bank is always exposed to prepayment risk.*

Now, since the lending rate $\alpha = \alpha(T)$ should be determined such that the present values of the both sides are equal, we set $T_R = F$ in (1) to obtain the following. In what follows, we assume that the bank is default-free. Also, we take the reduced-form approach. The *cumulative* default process is denoted by $H(t) = \int_0^t h(s)\mathrm{d}s$, $t \geq 0$, and we shall use the notation $M(t) = R(t) + H(t)$ as the *risk-adjusted* discount process.

Proposition 2 *Suppose that, under the risk-neutral probability measure P, the default process is a Cox process with intensity $h(t)$. Then, the lending interest rate is given by*

$$\alpha(T) = \frac{1 - E[\mathrm{e}^{-M(T)}] - \int_0^T \zeta(u)\mathrm{d}u}{\sum_{j=1}^m E[\mathrm{e}^{-M(t_j)}]}. \qquad (3)$$

where

$$\zeta(u) = E\Big[h(u)\min\{\beta L(u)/F, 1\}\mathrm{e}^{-M(u)}\Big].$$

3 A Gaussian Model

We consider the following Gaussian model. That is, under P, suppose that

$$\frac{\mathrm{d}L}{L} = r(t)\mathrm{d}t + \sigma_L \mathrm{d}z_L,$$

$$\mathrm{d}r = a_r(m_r - r)\mathrm{d}t + \sigma_r \mathrm{d}z_r,$$

and $\mathrm{d}z_L(t)\mathrm{d}z_r(t) = \rho\mathrm{d}t$. Also, the hazard rate $h(t)$ is a deterministic function in time t and $F = \beta = 1$ for the sake of simplicity.

Proposition 3 *In the Gaussian model described above, the lending interest rate is given by*

$$\alpha(T) = \frac{1 - v(T)\mathrm{e}^{-H(T)} - \int_0^T h(t)\mathrm{e}^{-H(t)}(v(t) - p(t))\mathrm{d}t}{\int_0^T v(t)\mathrm{e}^{-H(t)}\mathrm{d}t},$$

where $L = L(0)$ and $v(t)$ denotes the default-free discount bond price with maturity t. The put option premium is given by

$$p(t) = v(t)\Phi(S_Y(t) - d(t)) - L\Phi(-d(t)),$$

where

$$S_Y^2(t) = S_R^2(t) + \sigma_L^2 t + 2\rho\frac{\sigma_L\sigma_r}{a_r}\left[t - \frac{1 - \mathrm{e}^{-a_r t}}{a_r}\right]$$

and

$$d(t) = \frac{\log[L/v(t)]}{S_Y(t)} + \frac{1}{2}S_Y(t).$$

It is well known that the put option premium decreases in the initial price L and increases in the volatility $S_Y(t)$. Also, the correlation coefficient ρ appears only in the volatility $S_Y(t)$. Since $S_Y(t)$ is increasing in ρ, the next result is immediate.

Proposition 4 *In the Gaussian model described above, the lending rate is decreasing in the initial value L of the collateral asset and increasing in the correlation coefficient ρ between the collateral asset and the default-free spot rate.*

Since

$$\frac{\partial S_Y^2(t)}{\partial \sigma_L} = 2\sigma_L t + 2\rho\frac{\sigma_r}{a_r}\left[t - \frac{1 - \mathrm{e}^{-a_r t}}{a_r}\right],$$

where σ_L is the volatility of the collateral asset, we then have the following.

Proposition 5 *In the Gaussian model described above, if $\rho \geq 0$ then the lending rate is increasing in the volatility σ_L. If $\rho < 0$, the lending rate is decreasing in σ_L if and only if*

$$\sigma_L \leq \Sigma_r \equiv \frac{-\rho\sigma_r}{a_r t}\left[t - \frac{1 - \mathrm{e}^{-a_r t}}{a_r}\right].$$

From Proposition 5, when $\rho < 0$, the lending interest rate is decreasing in the collateral volatility σ_L as far as it is less than Σ_r. This is interesting, because it indicates that a larger volatility does not always result in a higher interest.

Next, we consider the effect of the hazard rate. For this purpose, we assume that the hazard function is given by

$$h(t) = \lambda\gamma(t + \eta)^{\gamma - 1}, \quad t \geq 0.$$

Some numerical results will be shown in the presentation.