Real Options in a Duopoly Market with General Volatility Structure

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1 Introduction

This paper considers strategic entry decisions in a duopoly market when the underlying state variable follows a diffusion with volatility that depends on the current state variable. The extension to this case is more than marginal, since empirical studies have suggested that the volatility is indeed non-constant in real options practices. It is shown that, even in the extended model, three types of equilibria exist in the case of strategic substitution, as for the geometric Brownian case, when the revenue functions are linear. Also, the presence of strategic interactions may push a firm with cost advantage to invest earlier, and the firm value as well as the optimal threshold for the investment decision increases as the market uncertainty increases.

2 The Model

Consider two firms both having the possibility to make an irreversible investment that increases their profits. The revenue of each firm is uncertain depending on the future condition for the product. We describe the uncertainty by state variable $\{X_t\}$ on the probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. It satisfies the stochastic differential equation

$$\frac{\mathrm{d}X_t}{X_t} = \mu \mathrm{d}t + \sigma(X_t)\mathrm{d}z_t, \quad t \geq 0, \tag{1}$$

where $\{z_t\}$ denotes a standard Brownian motion. It is a diffusion with linear drift μx and diffusion coefficient $x\sigma(x)$. Let $\pi_{N_iN_j}(x)$ denote the revenue flow of firm i when $X_t = x$, $x \geq 0$, where $N_k = 0$ if firm k $\in \{i,j\}$ has not invested, $N_k = 1$ if firm $k \in \{i,j\}$ has invested. It is assumed that $\pi_{N_iN_j}(x)$ is strictly increasing and continuous in x. As to the relative magnitude relation between the revenue functions, we assume that $\pi_{10}(x) > \pi_{00}(x)$ and $\pi_{11}(x) > \pi_{01}(x)$ for all $x \geq 0$. On the other hand, the relative relation between $\pi_{10}(x)$ and $\pi_{11}(x)$ (also $\pi_{00}(x)$ and $\pi_{01}(x)$) depends on the market condition. $\pi_{10} > \pi_{11}$ is called the case of *strategic substitution*, $\pi_{10} < \pi_{11}$ is the case of *strategic complement*. The sunk cost to adopt the investment opportunity is assumed to be constant and equal to I_i, $i \in \{1,2\}$, where $I_1 < I_2$. That is, firm 1 has an advantage for the cost relative to firm 2. We call the firm that invests first the *leader* and the other firm the *follower*. The upper-case letters L and F are used to stand for the leader, the follower, respectively. The *optimal* stopping time for firm j is denoted by τ_j^n if firm j is $n \in \{\mathrm{L}, \mathrm{F}\}$. And we denote the optimal threshold for τ_i^k by $n_i^*(i \in \{1,2\},\ n \in \{\mathrm{L},\mathrm{F}\})$.

In order to discuss the maximization problem, we first consider the following problem: $h(x)$, define

$$V_\infty(x) = \max_{\tau \in \mathcal{T}} E^x\left[\mathrm{e}^{-r\tau}h(X_\tau)\right], \tag{2}$$

where $\mathcal{T}$ denotes the set of admissible strategies.

Proposition 2.1 *Suppose that $h(x)$ is increasing and convex in $x \geq 0$ with $h(0) = 0$. Then, under regularity conditions, the value function $V_\infty(x)$ defined in (2) is also increasing and convex in x with $V_\infty(0) = 0$. The value function is also increasing in volatility.*

We note that there exists a optimal threshold x^* for the state variable X_t such that $\tau^* = \inf\{t \geq 0 : X_t = x^*\}$ where τ^* denotes the optimal stopping time for (2). Since $X_{\tau^*} = x^*$, it follows from (2) that

$$V_\infty(x) = E^x\left[\mathrm{e}^{-r\tau^*}\right]h(x^*). \tag{3}$$

Hence, calculation of the value function $V_\infty(x)$ is reduced to obtaining the Laplace transform $E^x[\mathrm{e}^{-r\tau^*}]$ of the *first passage time* τ^* to the optimal threshold x^*.

Corollary 2.1 *Suppose that the function $h(x)$ is increasing and convex in $x \geq 0$ with $h(0) = 0$. Then, under regularity conditions, the Laplace transform $E^x[\mathrm{e}^{-r\tau^*}]$ of the first passage time τ^* to the optimal threshold x^* is also increasing and convex in x with $E^0[\mathrm{e}^{-r\tau^*}] = 0$. The Laplace transform is also increasing in volatility.*

It is well known that the value function (and the Laplace transform) satisfies the following (ordinary) differential equation:

$$\frac{x^2\sigma^2(x)}{2}V_\infty''(x) + \mu x V_\infty'(x) = rV_\infty(x) \tag{4}$$

with boundary conditions $V_\infty(0) = 0$, $V_\infty(x^*) = h(x^*)$, $V_\infty'(x^*) = h'(x^*)$.

3 A Duopoly Market

Suppose that the current time is $t = 0$, that the leader j has already invested, i.e. $\tau_j^{\mathrm{L}} < 0$, and that the current state variable is below the critical value F_i^*. The last assumption is equivalent to saying that the follower i will not invest now. The value of firm j as the leader and that of firm i as the follower are given by

$$V_j^{\mathrm{L}}(x) = \frac{\pi_{10}x}{r-\mu} - I_j + E^x\left[\mathrm{e}^{-r\tau_i^{\mathrm{F}}}\right]\frac{(\pi_{11}-\pi_{10})\mathrm{F}_i^*}{r-\mu} \tag{5}$$

$$V_i^{\mathrm{F}}(x) = \frac{\pi_{01}x}{r-\mu} + E^x\left[\mathrm{e}^{-r\tau_i^{\mathrm{F}}}\right] \times \left\{\frac{(\pi_{11}-\pi_{01})\mathrm{F}_i^*}{r-\mu} - I_i\right\}. \tag{6}$$

4 Equilibria

We consider the case of strategic substitution only, i.e. $\pi_{11} < \pi_{10}$ for all $x > 0$. Now, from Corollary 2.1, it is readily seen that the value function $V_i^{\mathrm{F}}(x)$ in (6) is convex in x. Also, it must satisfy the value-matching condition and the smooth-pasting condition, i.e.

$$V_i^{\mathrm{F}}(\mathrm{F}_i^*) = \frac{\pi_{11}\mathrm{F}_i^*}{r-\mu} - I_i, \quad \left(V_i^{\mathrm{F}}(\mathrm{F}_i^*)\right)' = \frac{\pi_{11}}{r-\mu}, \tag{7}$$

respectively. the function $V_i^{\mathrm{F}}(x)$ must satisfy the smooth-pasting condition, because it is determined by optimizing the stopping time τ_i^{F}.

On the other hand, the value function $V_j^{\mathrm{L}}(x)$ in (5) is concave in x and satisfies the value-matching condition

$$V_j^{\mathrm{L}}(\mathrm{F}_i^*) = \frac{\pi_{11}\mathrm{F}_i^*}{r-\mu} - I_j. \tag{8}$$

Note that the function $V_j^{\mathrm{L}}(x)$ does not satisfy the smooth-pasting condition, because it is not optimized in terms of the stopping time τ_i^{F}.

Since it is easlily seen that $\mathrm{F}_1^* < \mathrm{F}_2^*$ from some caluculation, Figure 1 depicts the value functions V_2^{F} and V_2^{L} for firm 2 in some case of strategic substitution. Then, there is a preemption equilibrium. That

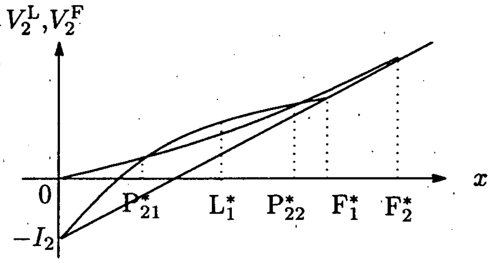

Figure 1: the value function for firm 2 in some case

is, if $\mathrm{L}_1^* \in [\mathrm{P}_{21}^* \mathrm{P}_{22}^*]$ where $\mathrm{P}_{21}^* = \inf\{x \geq 0 : V_2^{\mathrm{L}} \geq V_2^{\mathrm{F}}\}$, $\mathrm{P}_{22}^* = \sup\{x \geq 0 : V_2^{\mathrm{L}} \geq V_2^{\mathrm{F}}\}$ in Figure 1, firm 2 may have an incentive to preepmt firm 1. Hence, firm 1 invests at time τ_{21}^{P} where $\tau_{21}^{\mathrm{P}} = \inf\{t \geq 0 : X_t = \mathrm{P}_{21}^*\}$ with taking account into firm 2's strategy, while firm 2 invests at time τ_2^{F}.

On the other hands, there does not exits any preemption equilibrium for the case of strategic complement.

References

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