VaR is Subject to a Significant Positive Bias

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1 Introduction

Value-at-Risk (VaR) measures the potential for significant loss in a portfolio of financial assets. In the context of statistics, VaR is defined as a sample quantile of the portfolio loss distribution.

In this article, we show that VaR estimators commonly used by practitioners have a strong positive bias. The bias increases as the confidence level increases, the degree of fat-tailness increases, and the number of samples decreases.

2 Some Preliminaries

We fix a probability space $(\Omega, \mathcal{F}, P)$ and consider a real-valued random variable X that represents the profit/loss of a portfolio over a given risk horizon. The cumulative distribution function (CDF for short) of X is denoted by $F_X(x)$, $x \in \boldsymbol{R}$, i.e. $F_X(x) = P\{X \leq x\}$, where $\boldsymbol{R}$ stands for the real line. It is assumed throughout that the CDF $F_X(x)$ is absolutely continuous with probability density function (PDF for short) denoted by $f_X(x) > 0$ for $x \in \boldsymbol{R}$.

Definition 1 (α Quantile and VaR) For each α, $0 < \alpha < 1$, the α-quantile of $F_X(x)$ is given by

$$F_X^{-1}(\alpha) = \inf\{x | F_X(x) \geq \alpha\}. \qquad (1)$$

VaR with $100(1-\alpha)\%$ confidence level is defined by

$$\mathrm{VaR}_{(1-\alpha)} = -F_X^{-1}(\alpha) = -\inf\{x | F_X(x) > \alpha\}. \quad (2)$$

2.1 Empirical Distribution and VaR

Suppose that we sample n observations from X independently. The samples are denoted by $X_1, X_2, \ldots, X_n$. Note that they are IID samples with CDF $F_X(x)$.

Definition 2 (Empirical Distribution) The empirical distribution generated from the samples $(X_1, X_2, \ldots, X_n)$ is defined by

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{x \geq X_i\}}, \quad x \in \boldsymbol{R}, \qquad (3)$$

where $\mathbf{1}_A$ denotes the indicator function.

We note that the empirical distribution $F_n(x)$ depends on the samples $(X_1, X_2, \ldots, X_n)$, i.e. it is an estimator of the true CDF $F_X(x)$. It is an *unbiased* estimator, since

$$E[F_n(x)] = \frac{1}{n}\sum_{i=1}^{n} E[\mathbf{1}_{\{x \geq X_i\}}] = F_X(x).$$

Also, it is well known that $F_n(x)$ converges weakly to $F_X(x)$ as $n \to \infty$.

Definition 3 (VaR Estimator) For given α and samples $(X_1, X_2, \ldots, X_n)$, let k be such that $k/n \leq \alpha < (k+1)/n$. Then, the VaR estimator with $100(1-\alpha)\%$ confidence level is defined by

$$\mathrm{VaR}_{(1-\alpha)} = X_{k:n}, \quad \frac{k}{n} \leq \alpha < \frac{k+1}{n}, \qquad (4)$$

where $X_{k:n}$ denotes the kth order statistics of the samples. In particular, if $n\alpha$ is an integer, then the VaR estimator is given by $\mathrm{VaR}_{(1-\alpha)} = X_{n\alpha:n}$.

One of the most popular smoothing methods for quantile estimates is to use a linear combination of order statistics $X_{i:n}$, called *L-estimators*. That is, an L-estimator is given by $\sum_{i=1}^{n} w_i X_{i:n}$ for some $w_i \geq 0$ with $\sum_{i=1}^{n} w_i = 1$. Note that the VaR estimator given by (4) can be regarded as an L-estimator with $w_k = 1$ and $w_i = 0$ for $i \neq k$.

Definition 4 (Harrell–Davis Estimator) For each $\alpha \in (0,1)$, let k be the smallest integer not less than $n\alpha$. Then, the Harrell–Davis estimator is defined as

$$\mathrm{HD}_\alpha = \sum_{i=1}^{n} w_{i:n}^{\alpha} X_{i:n}, \tag{5}$$

$$w_{i:n}^{\alpha} = \frac{1}{\beta(k, n-k+1)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} y^{k-1}(1-y)^{n-k} \mathrm{d}y.$$

Note that the weights $w_{i:n}^{\alpha}$ depend only on the parameters n and α. In other words, no matter what the samples X_i are, the weights $w_{i:n}^{\alpha}$ are calculated for each $i = 1, 2, \ldots, n$, once the percentile α and the sample size n are given. Harrell–Davis estimator is exactly the same as the bootstrap estimator.

3 Theoretical Results

We shall show below that $E[X_{k:n}]$ converges to $F_X^{-1}(\alpha)$ from below as $n \to \infty$. Hence, the VaR estimator is biased positively for any finite sample.

Definition 5 (Concave Ordering) For two random variables X and Y, X is said to be greater than Y in the sense of concave ordering, denoted by $X \geq_{\mathrm{cv}} Y$, if $E[h(X)] \geq E[h(Y)]$ for any concave function $h(x)$, for which the expectations exist.

The next result shows that the sequence of random variables $\{Z_n\}$ is monotonically increasing in the sense of concave ordering.

Proposition 1 *Suppose that the random variable Z_n follows the β-distribution $\beta(k, n-k+1)$ with $k = (n+1)\alpha$. Then, $Z_{n+1} \geq_{\mathrm{cv}} Z_n$ for every $n = 1, 2, \ldots$.*

It should be noted that the CDF $F_X(x)$ with a bell-shaped tail, such as normal distributions and t-distributions, is convex in the interval $(-\infty, x^*)$ for some x^*. Hence, its inverse function is concave in the interval $(0, y^*)$ for some small y^*.

Next, we consider a bias of the Harrell–Davis estimator. From (5), we obtain

$$E[\mathrm{HD}_\alpha] = \sum_{i=1}^{n} w_{i:n}^{\alpha} E[X_{i:n}], \tag{6}$$

since the weights $w_{i:n}^{\alpha}$ are independent of the samples. From proposition 1, we can show that $E[\mathrm{HD}_\alpha] < E[X_{k:n}]$, where $k = (n+1)\alpha$, under some mild conditions. Hence, the Harrell–Davis estimator, or its simulation counterpart bootstrap estimator, has more bias than the ordinary quantile estimator in certain circumstances.

4 Simulation Results

It is explicitly observed that both the VaR and the Harrell–Davis estimators have positive biases. The bias increases as the confidence level increases, the DF of the underlying t-distribution decreases (the degree of fat-tailness increases), and the number of samples decreases. Also, in the case of 99% confidence level, the Harrell–Davis estimator tends to have a more bias except the small sample case. In the case of 95% confidence level, the Harrell–Davis bias is similar to that of the quantile VaR estimator. These observations are consistent with the theoretical results discussed in the previous section.

References

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