Exact Formulation of Stochastic EMQ Model for an Unreliable Production System

Bibhas C. GIRI and Tadashi DOHI (01307065)
Department of Information Engineering, Hiroshima University

1. Summary

EMQ models with stochastic machine breakdown and repair have been developed in the literature assuming negligible corrective repair time and without preventive maintenance (PM) (Groenevelt *et al.*[1]), constant corrective repair time and without PM (Kim *et al.*[2]), generally distributed corrective repair time and without PM (Abboud[3]). EMQ model without PM or PM with negligible time can not provide appropriate production-maintenance strategy. The purpose of this paper is to present an exact formulation of stochastic EMQ model for an unreliable production system under a general framework in which the time to failure, corrective and preventive repair times are taken as random variables. The criteria for the existence and uniqueness of the optimal production time (lot size) are derived under arbitrary as well as specific failure and repair time distributions.

2. The general model

Notation:

X: non-negative i.i.d. random variable denoting time to machine failure; $F_X(t)$: failure time distribution with p.d.f. $f_X(t) = dF_X(t)/dt$; $G_1(l_1)$: corrective repair time distribution with p.d.f. $g_1(l_1) = dG_1(l_1)/dl_1$; $G_2(l_2)$: preventive repair time distribution with p.d.f. $g_2(l_2) = dG_2(l_2)/dl_2$; p (> 0): production rate; d $(< p)$: demand rate $C_0(> 0)$: set up cost; $C_1(> 0)$: corrective repair co t per unit time; $C_2(< C_1)$: preventive repair cost per unit time; $C_i(> 0)$: holding cost per unit product per unit time; $C_s(> 0)$: shortage cost per unit product; Q (> 0): order quantity.

Description and formulation of the model:

Consider a single-unit production system which starts at time $t = 0$ with the aim of producing an amount Q. If the machine does not fail until time $t = Q/p$ then the process is stopped and PM is carried out to return back the machine to the same initial working condition before the start of the next production cycle. If, however, the machine fails before producing Q units then the corrective repair action is started immediately. During machine repair, the demand is met first from the accumulated inventory. If there is sufficient stock to meet the demand during machine repair then the next production starts when the on-hand inventory is exhausted. If shortages occur, they are not delivered after machine repair and are considered lost. In order to avoid an unrealistic decision making, we assume $\underline{Q} \le Q \le \overline{Q}$, where $\underline{Q}$ and $\overline{Q}$ are lower and upper limits of Q respectively. The mean time length of a cycle and the expected total cost for one cycle are given by

$$T(Q) = \int_0^\infty \mathrm{E}[\text{duration of a cycle} \mid X = t]\, f_X(t)dt$$

$$= \int_0^{\frac{Q}{p}} \left[\int_0^{\frac{(p-d)t}{d}} \frac{pt}{d}\, dG_1(l_1) + \int_{\frac{(p-d)t}{d}}^\infty (t + l_1)\, dG_1(l_1) \right] \times dF_X(t) + \int_{\frac{Q}{p}}^\infty \left[\int_0^{\frac{(p-d)Q}{pd}} \frac{Q}{d}\, dG_2(l_2) + \int_{\frac{(p-d)Q}{pd}}^\infty (Q/p + l_2)\, dG_2(l_2) \right] dF_X(t),$$

$$S(Q) = \int_0^\infty \mathrm{E}[\text{cost per cycle} \mid X = t]\, f_X(t)dt$$

$$= C_0 + C_1 \int_0^{Q/p} \int_0^\infty l_1\, dG_1(l_1)\, dF_X(t) + C_2 \int_{Q/p}^\infty \int_0^\infty l_2\, dG_2(l_2)\, dF_X(t) + C_i \left[\int_0^{\frac{Q}{p}} \frac{(p-d)pt^2}{2d} dF_X(t) + \int_{\frac{Q}{p}}^\infty \frac{(p-d)Q^2}{2pd} dF_X(t) \right]$$
$$+ C_s d \int_0^{\frac{Q}{p}} \int_{\frac{(p-d)t}{d}}^\infty \left\{ l_1 - \frac{(p-d)t}{d} \right\} dG_1(l_1)\, dF_X(t) + C_s d \int_{\frac{Q}{p}}^\infty \int_{\frac{(p-d)Q}{pd}}^\infty \left\{ l_2 - \frac{(p-d)Q}{pd} \right\} dG_2(l_2) dF_X(t),$$

respectively. By the well-known renewal reward theorem, the expected cost per unit time in the steady state is given by

$$C(Q) = \lim_{t\to\infty} \frac{\mathrm{E}[\text{total cost on } (0,t]]}{t} = \frac{S(Q)}{T(Q)}. \quad (1)$$

Then the problem is to find the optimal production lot size Q^* which minimizes $C(Q)$, subject to $\underline{Q} \le Q^* \le \overline{Q}$. Let $t_0 = Q/p, \underline{t_0} = \underline{Q}/p$ and $\overline{t_0} = \overline{Q}/p$. Define the numerator of the derivative of $C(t_0) = S(t_0)/T(t_0)$ w.r.t. t_0, divided by $1 - F_X(t_0)$ as $q(t_0)$, *i.e.*,

$$q(t_0) = \left[(p-d) \left\{ \frac{C_i p t_0}{d} - C_s \overline{G_2} \left(\frac{(p-d)t_0}{d} \right) \right\} + r_X(t_0) \left\{ C_1 m_1^{-1} - C_2 m_2^{-1} + C_s d \left(m_1^{-1} - m_2^{-1} \right) + C_s d \right. \right.$$

$$\times \int_0^{\frac{(p-d)t_0}{d}} G_1(l_1)dl_1 - C_s d \int_0^{\frac{(p-d)t_0}{d}} G_2(l_2)dl_2 \Big\}\Big] T(t_0)$$

$$- \Big[1 + r_X(t_0)\Big\{(m_1^{-1} - m_2^{-1}) + \int_0^{\frac{(p-d)t_0}{d}} G_1(l_1)dl_1$$

$$- \int_0^{\frac{(p-d)t_0}{d}} G_2(l_2)dl_2\Big\} + \frac{(p-d)}{d} G_2\left(\frac{(p-d)t_0}{d}\right)\Big] S(t_0)$$

where m_1^{-1} $(>0) = \int_0^\infty l_1\ dG_1(l_1)$; m_2^{-1} $(>0) = \int_0^\infty l_2\ dG_1(l_2)$, and $r_X(t) = f_X(t)/\left(1 - F_X(t)\right)$.

Differentiating $q(t_0)$ with respect t_0, we get

$$\frac{dq(t_0)}{dt_0} = \phi(t_0)T(t_0) + \psi(t_0)[C_s d\ T(t_0) - S(t_0)], \text{where}$$

$$\phi(t_0) = \frac{dr_X(t_0)}{dt_0}\left(C_1 m_1^{-1} - C_2 m_2^{-1}\right) + \frac{C_i p(p-d)}{d},$$

$$\psi(t_0) = \frac{dr_X(t_0)}{dt_0}\Big\{m_1^{-1} - m_2^{-1} + \int_0^{\frac{(p-d)t_0}{d}} G_1(l_1)dl_1$$

$$- \int_0^{\frac{(p-d)t_0}{d}} G_2(l_2)dl_2\Big\} + \frac{(p-d)^2}{d^2} g_2\left(\frac{(p-d)t_0}{d}\right)$$

$$+ \frac{(p-d)}{d} r_X(t_0)\Big\{G_1\left(\frac{(p-d)t_0}{d}\right) - G_2\left(\frac{(p-d)t_0}{d}\right)\Big\}.$$

To derive the criteria for the existence and uniqueness of the optimal production time t_0^* we make the following assumptions:

(A-1) $\phi(\overline{t_0}) > 0$.

(A-2) $G_1(\cdot)$ and $G_2(\cdot)$ $[G_1 > G_2]$ are both continuous and increasing functions in the interval $[0, (p-d)\underline{t_0}/d]$ such that $\psi(\underline{t_0}) \geq 0$.

(A-3) The opportunity loss per unit demand $C_s d$ and the long-run average cost $C(t_0)$ are such that $C_s d < C(t_0) < C_s d + \phi(\overline{t_0})/\psi(\underline{t_0})\ \forall\ t_0 \in [\underline{t_0}, \overline{t_0}]$.

Theorem 1: Suppose that the failure time distribution $F_X(t)$ is IFR (Increasing Failure Rate). Under assumptions (A-1)-(A-3), (i) if $q(\underline{t_0}) < 0$ and $q(\overline{t_0}) > 0$ then there exists a finite and unique optimal production time t_0^* $(0 < \underline{t_0} \leq t_0^* \leq \overline{t_0} < \infty)$ satisfying the non-linear equation $q(t_0^*) = 0$.

(ii) If $q(\overline{t_0}) \leq 0$ then $t_0^* = \overline{t_0}$ and if $q(\underline{t_0}) \geq 0$ then $t_0^* = \underline{t_0}$.

3. Special case

Suppose that $f_X(t) = \lambda \exp\{-\lambda t\}$, $\lambda > 0$; $g_1(l_1) = 1/b_1$, $0 \leq l_1 \leq b_1$; $g_2(l_2) = 1/b_2$, $0 \leq l_1 \leq b_1$, where $b_1, b_2 > 0$. Then, after some algebraic manipulations, the mean time length of one cycle is obtained as

$$T_1(t_0) = \frac{p(p-d)}{d^2 b_1}\left[\frac{2 - 2e^{-\lambda t_0}}{\lambda^2} - \left(\frac{2t_0}{\lambda} + t_0^2\right)e^{-\lambda t_0}\right]$$

$$+ \frac{1}{b_1}\Big[\frac{b_1^2}{2}\left(1 - e^{-\lambda t_0}\right) - \Big\{b_1 t_0 - \frac{(p-d)t_0^2}{d} - \frac{(p-d)^2 t_0^2}{2d^2}\Big\}$$

$$\times e^{-\lambda t_0} - \Big\{b_1 - \frac{2(p-d)t_0}{d} - \frac{(p-d)^2 t_0}{d^2}\Big\}\frac{e^{-\lambda t_0}}{\lambda} + \frac{b_1}{\lambda}$$

$$+ \Big\{\frac{2(p-d)}{d} + \frac{(p-d)^2}{d^2}\Big\}\frac{e^{-\lambda t_0} - 1}{\lambda^2}\Big] + \frac{p(p-d)t_0^2}{d^2 b_2} e^{-\lambda t_0}$$

$$+ \frac{1}{b_2}\left[b_2 t_0 - \frac{(p-d)t_0^2}{d} + \frac{b_2^2}{2} - \frac{(p-d)^2 t_0^2}{2d^2}\right]e^{-\lambda t_0}, \qquad (2)$$

and the expected total cost per cycle is

$$S_1(t_0) = C_0 + \frac{C_1 b_1}{2}\left(1 - e^{-\lambda t_0}\right) + \frac{C_2 b_2}{2} e^{-\lambda t_0}$$

$$+ \frac{C_i(p-d)p}{2d}\left[\frac{2}{\lambda^2}\left(1 - e^{-\lambda t_0}\right) - \frac{2t_0}{\lambda} e^{-\lambda t_0}\right] + \frac{C_s d\lambda}{2b_1}\Big[b_1^2$$

$$- \Big\{b_1 - \frac{(p-d)t_0}{d}\Big\}^2 e^{-\lambda t_0} + \frac{2(p-d)}{d\lambda}\Big\{b_1 - \frac{(p-d)t_0}{d}\Big\}$$

$$\times e^{-\lambda t_0} - \frac{2(p-d)b_1}{d\lambda} - \frac{2(p-d)^2}{d^2\lambda^2}\left(e^{-\lambda t_0} - 1\right)\Big] + \frac{C_s d}{2b_2}$$

$$\times \left[b_2 - \frac{(p-d)t_0}{d}\right]^2 e^{-\lambda t_0}. \qquad (3)$$

Define the numerator of the derivative of $C_1(t_0) = S_1(t_0)/T_1(t_0)$ w.r.t. t_0 divided by $\exp\{-\lambda t_0\}$ as $q_1(t_0)$. Then

$$q_1(t_0) = \Big[\frac{\lambda}{2}(C_1 b_1 - C_2 b_2) + \frac{C_i p(p-d)t_0}{d} + \frac{C_s d\lambda}{2b_1}\Big\{b_1$$

$$- \frac{(p-d)t_0}{d}\Big\}^2 - \frac{C_s d(p-d)}{b_2}\Big\{b_2 - \frac{(p-d)t_0}{d}\Big\} - \frac{C_s d\lambda}{2b_2}$$

$$\times \Big\{b_2 - \frac{(p-d)t_0}{d}\Big\}^2\Big] T_1(t_0) - \Big[\frac{\lambda(p-d)^2}{2d^2}\left(\frac{1}{b_1} - \frac{1}{b_2}\right)t_0^2$$

$$+ \frac{(p-d)^2 t_0}{d^2 b_2} + 1 + \frac{\lambda}{2}(b_1 - b_2)\Big] S_1(t_0). \qquad (4)$$

Theorem 2: Suppose that

(A-4) $\left(b_1^{-1} - b_2^{-1}\right)\lambda\underline{t_0} + b_2^{-1} \leq 0$ and $C_1(t_0) > C_s d$.

(A-5)(a) $\left(b_1^{-1} - b_2^{-1}\right)\lambda\overline{t_0} + b_2^{-1} \geq 0$, (b) $C_s d < C_1(t_0) < C_s d + C_i pd/\left[(p-d)\left\{(b_1^{-1} - b_2^{-1})\lambda\underline{t_0} + b_2^{-1}\right\}\right]$.

Under (A-4) or (A-5), (i) if $q_1(\underline{t_0}) < 0$ and $q_1(\overline{t_0}) > 0$ then the unique optimal production time t_0^* exists in the interval $(\underline{t_0}, \overline{t_0})$ satisfying $q_1(t_0^*) = 0$.

(ii) if $q_1(\underline{t_0}) \geq 0$ then t_0^* is $\underline{t_0}$. On the other hand, if $q_1(\overline{t_0}) \leq 0$ then t_0^* is $\overline{t_0}$.

Remark: We have not considered more general failure distribution *e.g.* Weibull distribution or Gamma distribution *etc.* instead of exponential distribution because of mathematical intractibility.

References

1. Groenevelt H, Pintelon L and Seidmann A (1992). Production lot sizing with machine breakdowns. *Mgmt Sci* **38**: 104-123.
2. Kim CH, Hong, Y and Kim, SY (1997). An extended optimal lot sizing model with an unreliable machine. *Prod Plan & Cont* **8**: 577-85.
3. Abboud NE (1997). A simple approximation of the EMQ model with Poisson machine failures. *Prod Plan & Cont* **8**: 385-397.