Home-Away Table Feasibility Problem

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1. Introduction

Constructing an appropriate schedule for a sports competition is an important task for the organizers of the competition because a schedule substantially affects the results of games. In this manuscript, we consider a problem arising from scheduling of a round-robin tournament with home-away assignment.

A *round-robin tournament* is a tournament in which each team matches in turn against every other team. We deal with a round-robin tournament consisting of $2n$ teams with $2n-1$ slots; each team plays just one game in each slot. Each team has its home, and each game is held at the home of one of the teams playing. A game between teams t_1 and t_2 played at the home of t_1 is called a *home game* for t_1 and an *away game* for t_2. In the schedule of Fig. 1, a game with '@' means that the game is an away game and one without '@' is a home game for the team corresponding to the row. For example, team 4 plays against team 3 at the home of team 4 in slot 5.

A *Home-Away Table* (HAT) is a table showing whether each team plays at home or at away in each slot. Figure 2 is the HAT corresponding to the schedule of Fig. 1. In each slot of a HAT, each 'H' means that the team plays at home, while each 'A' means at away.

Assume that we need to construct a schedule from a given HAT. If a HAT has at least one corresponding schedule, we say that the HAT is *feasible*. Unfortunately, there is a HAT that cannot generate a schedule; such a HAT is called an *infeasible* HAT. The *HAT feasibility problem* is to determine the feasibility of a given HAT.

The HAT feasibility problem is a long-standing question in sports scheduling [2], and a problem of significance in practice. For this problem, polynomial-size characterization of feasible HATs has not been found yet, and whether this problem is NP-complete or not is still open.

In the next section, we introduce a special class of HATs and previous results on the HAT feasibility problem.

2. Equitable HAT and Previous Results

In practical sports scheduling, a tournament organizer often prefers a HAT satisfying particular properties; such a HAT is called an

(team)		1	2	3	4	5 (slot)
1	:	5	3	@2	4	@6
2	:	4	@6	1	3	@5
3	:	6	@1	5	@2	@4
4	:	@2	@5	6	@1	3
5	:	@1	4	@3	@6	2
6	:	@3	2	@4	5	1

Figure 1: Schedule of six teams

(team)		1	2	3	4	5 (slot)
1	:	H	H	A	H	A
2	:	H	A	H	H	A
3	:	H	A	H	A	A
4	:	A	A	H	A	H
5	:	A	H	A	A	H
6	:	A	H	A	H	H

Figure 2: HAT corresponding to Fig. 1

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equitable HAT. An equitable HAT is a HAT in which each team has just one consecutive 'H's or 'A's. In fact, Fig. 2 is an equitable HAT of six teams.

Here we mention the previously obtained results on the feasibility of an equitable HAT, described in [1]. In the remainder, U denotes a set of teams of a given HAT, i.e. $\{1, 2, \ldots, 2n\}$. Let H and A be functions that take $T \subseteq U$ and slot $s \in \{1, 2, \ldots, 2n-1\}$ as arguments, and return the number of 'H's and that of 'A's in s among T, respectively. Then, the following condition is a necessary condition for a feasible HAT.

$$\forall T \subseteq U,\ \sum_{s=1}^{2n-1} \min\{H(T,s),\ A(T,s)\} \geq {}_{|T|}\mathrm{C}_2 \tag{1}$$

In addition, we have shown the following by computational experiments with integer programming: Condition (1) is a sufficient condition for a feasible equitable HAT when the number of teams is up to 26. (For more than 26 teams, we did not perform computational experiments.) We have conjectured that, for any number of teams, Condition (1) is a necessary and sufficient condition for a feasible equitable HAT. This conjecture is still open.

Although Condition (1) is much powerful for deciding the feasibility of an equitable HAT, to check whether a given equitable HAT satisfies Condition (1) took exponential steps. In the next section, we give the theorems solving this difficulty.

3. Main Theorem

We proved the following theorem.

Theorem 1. *Whether an equitable HAT satisfies Condition (1) or not can be examined in polynomial steps.*

We describe this theorem more precisely. Since permutation of the rows of a HAT does not change the feasibility, we may assume that an equitable HAT satisfies the following:

- teams $1, 2, \ldots, n$ have 'H' in slot 1;
- team t has consecutive 'H's or 'A's in slots earlier than the slots in which team $t+1$ has consecutive 'H's or 'A's ($t = 1, 2, \ldots, n-1$).

It should be noted that these assumption does not lose generality of an equitable HAT [3].

Theorem 2. *For an equitable HAT of any number of teams, the following holds:*

$$\forall T \subseteq U,\ \sum_{s=1}^{2n-1} \min\{H(T,s), A(T,s)\} \geq {}_{|T|}\mathrm{C}_2$$

$$\Longleftrightarrow$$

$$\forall T \subseteq U \text{ s. t. } |T| \leq n,\ T \text{ is consecutive},$$
$$\sum_{s=1}^{2n-1} \min\{H(T,s), A(T,s)\} \geq {}_{|T|}\mathrm{C}_2.$$

By Theorem 2, the number of team subsets to be checked is reduced to $O(n^2)$ and consequently Theorem 1 holds.

These results provide a highly efficient algorithm for finding feasible equitable HATs of practical size.

References

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[2] D. de Werra: Geography, Games and Graphs. Discrete Applied Mathematics, **2** (1980), pp. 327–337.

[3] R. Miyashiro, H. Iwasaki and T. Matsui: Characterizing Feasible Pattern Sets with a Minimum Number of Breaks. **METR 2002-14**, Technical Report, Department of Mathematical Informatics, Graduate School of Information Science and Technology, the University of Tokyo.

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