# Mathematical Properties of a DEA Game

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## 1. Introduction

In the problem of evaluating the performance among individuals or organizations with multiple criteria, 'variable weight,' which is newly used in data envelopment analysis (DEA), is an epoch-making concept contrasted to *a priori* 'fixed weight.' In this concept, each decision making unit (DMU) may choose a set of weights to the criteria that are most preferable to the DMU, i.e., the weights to the criteria may differ from one DMU to another. However, we might be faced with a difficult problem of adjusting the difference of weights for the consensus-making among DMUs. In order to analyze such a situation, we constructed a DEA game using cooperative game theory and DEA [1, 2]. This paper uncovers some of its mathematical properties.

## 2. Basic models of the game

### 2.1 A DEA maximum game

Let $X = (x_{ij}) \in R_{+}^{m\times n}$ be the score matrix, consisting of the record $x_{ij}$ of player $j$ to the criterion $i$. For a coalition $S$ which is a subset of the player set $N = (1, \cdots, n)$, a characteristic function value $c(S)$ is defined as follows:

$$c(S) = \max_{w_i} \sum_{i=1}^{m} w_i (\sum_{j\in S} x_{ij}) \tag{1}$$

$$\text{subject to } \sum_{i=1}^{m} w_i (\sum_{j=1}^{n} x_{ij}) = 1$$

$$w_i \geq 0 \; (\forall i),$$

where $w_i$ is the weight assigned to the criterion $i$. Thus, we have a game in coalition form as represented by $(N, c)$. We have the following two propositions.

**Proposition 1**

$$\sum_{j=1}^{n} c(j) \geq 1. \tag{2}$$

**Proposition 2**

$$c(N) = 1. \tag{3}$$

### 2.2 A DEA minimum game

We define a game $(N, d)$ by replacing *max* in (1) by *min* as follows:

$$d(S) = \min_{w_i} \sum_{i=1}^{m} w_i (\sum_{j\in S} x_{ij}) \tag{4}$$

$$\text{subject to } \sum_{i=1}^{m} w_i (\sum_{j=1}^{n} x_{ij}) = 1$$

$$w_i \geq 0 \; (\forall i).$$

Similarly to the DEA *max* game, we have the following two propositions.

**Proposition 3**

$$1 \geq \sum_{j=1}^{n} d(j). \tag{5}$$

**Proposition 4**

$$d(N) = 1. \tag{6}$$

### 2.3 The relationship between two games

The DEA *min* game can be interpreted as the opposite of the *max* game. In one game each player is supposed to be selfish or bullish, whereas he/she is supposed to be modest or bearish in another game. However, within the framework of the DEA-games, between the games $(N, c)$ and $(N, d)$ we have the following proposition.

**Proposition 5**

$$d(S) + c(N - S) = 1, \; \forall S \subset N. \tag{7}$$

### 2.4 Additive property

We have the followong two propositions.

**Proposition 6** *The DEA max game (N, c) is sub-additive.*

**Proposition 7** *The DEA min game (N, d) is super-additive.*

Using the characteristic function $c$, the another game $(N, v)$ is defined by

$$v(S) = \sum_{j \in S} c(j) - c(S). \tag{8}$$

Then, the game $(N, v)$, which has essentially the same structure as the game $(N, c)$, is super-additive.

### 2.5 Convexity or concavity

The game $(N, c)$ is not always a concave game because we have a counterexample. However, this concept, concavity, is case-sensitive and so we should check it case by case. In the case of $S \cup T = N$, we have the following proposition.

**Proposition 8**

$c(S) + c(T) \geq 1 + c(S \cap T)$, $\forall S, T \subset N$ *with* $S \cup T = N$.

Similarly we have, for the game $(N, d)$,

**Proposition 9**

$1 + d(S \cap T) \geq d(S) + d(T)$, $\forall S, T \subset N$ *with* $S \cup T = N$.

From Proposition 8 and 9, we have the following two propositions in the case of 3 players.

**Proposition 10** *The DEA max game (N, c) with 3 players is concave.*

**Proposition 11** *The DEA min game (N, d) with 3 players is convex.*

## 3 The solution of the game

### 3.1 The core

Owen [3] has introduced linear program games associated with an economic production process and demonstrated that they have a non-empty core. We found that the DEA game can be interpreted as the dual of Owen's LP game. As a special case of his game, we have the following two propositions.

**Proposition 12** *The DEA min game (N, d) is a balanced game. Hence its core is non-empty.*

**Proposition 13** *The transformed game (N, v) is a balanced game. Hence its core is non-empty.*

Furthermore, with reference to Owen's study, we have the following proposition.

**Proposition 14** *For any* $w=(w_1, \cdots, w_m) \in R^m_+$ *in the simplex* $w_1 + \cdots + w_m = 1$*, the vector* $wX$ *is an imputation in the core of the DEA game.*

However, the reverse is not always true, i.e., there are imputations in the core that cannot be expressed as $wX$.

### 3.2 On the dominance relationship

We call player A *dominates* player B if it holds $x_{iA} \geq x_{iB}$ $(\forall i)$. We have the following dominance relationship on imputation.

**Proposition 15** *If player A dominates player B, then the Shapley value of A is not less than that of player B.*

**Proposition 16** *If player A dominates player B, then the nucleolus of A is not less than that of player B.*

### 3.3 One and the same Shapley value

Regarding the Shapley values of the DEA-games, we have the following property.

**Proposition 17** *The Shapley values of the DEA max and min games (N, c) and (N, d) are the same.*

This Proposition 17 might be one of the remarkable characteristics of the DEA game. The nucleolus of the DEA *min* game is not necessarily the same as that of the *max* game. The Shapley solution has strong impact on consensus-making among participants of the DEA game.

From the additivity axiom of the Shapley value, we have the following proposition.

**Proposition 18** *For any* $t \in R$*, the Shapley value of the game (N, tc+(1-t)d) is the same as that of the games (N, c) and (N, d).*

### 3.4 The Shapley value in the core

It is proved that the Shapley value of a convex game belongs to the core of the game. Hence, by Proposition 11, the Shapley value of the DEA *min* game $(N, d)$ for the 3-player case belongs to the core. Although the DEA *min* game with 4 or more players is not necessarily convex, we have the following proposition.

**Proposition 19** *The Shapley value of the DEA min game (N, d) with 4 players is included in the core.*

# References


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[3] Owen, G., (1975), "On the linear production games," *Mathematical Programming*, 9, 358-370.