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A Cutting Plane Approach to Hub Network Design Problems

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1. Introduction

We consider a network design problem with hub-and-spoke structure which arises from the airline industry. The hub-and-spoke structure is a network such that some nodes, called non-hub nodes, can interact only via a set of completely interconnected nodes, called hub nodes. This paper deals with the case that the hub nodes and non-hub nodes are fixed. The problem is to decide connections of each non-hub node to one of the hub nodes minimizing a total transportation cost. Saito, Matuura, and Matsui proposed a mixed integer programming formulation of this problem [2]. In this paper, we adopt a polyhedral approach to this formulation.

2. Formulation

Let H and N be the sets of hub nodes and non-hub nodes, respectively, with $|H| = p$ and $|N| = n$. Let $A = N \times N$ be a set of non-hub pairs. For any pair of non-hubs $(i,j) \in A$, the flow from i to j is denoted by $w_{ij} \geq 0$. For any pair of nodes $(p,q) \in (H \times H) \cup (H \times N) \cup (N \times H)$, the unit transportation cost from p to q is denoted by $c_{pq} \geq 0$. We can formulate the problem as follows;

(HLP)

$$\begin{array}{ll} \text{min.} & \sum_{(i,j)\in A} w_{ij}(\sum_{k\in H} c_{ik}x_{ik} \\ & \quad + \sum_{k\in H}\sum_{l\in H} c_{kl}x_{ik}x_{jl} + \sum_{l\in H} c_{lj}x_{jl}) \\ \text{s. t.} & \sum_{k\in H} x_{ik} = 1 \quad (\forall i \in N), \\ & x_{ik} \in \{0,1\} \quad (\forall (i,k) \in N \times H). \end{array}$$

By introducing new variables $y_{ikjl} = x_{ik}x_{jl}$, we can linearize the quadratic objective function. Let $\mathcal{A}$ be an affine subspace defined by the following equations:

$$\begin{cases} \sum_{k\in H} x_{ik} = 1 \quad (\forall i \in N), \\ -x_{ik} + \sum_{l\in H} y_{ikjl} = 0 \\ \qquad\qquad (\forall i, \forall j \in N, \forall k \in H, i < j), \\ -x_{jl} + \sum_{k\in H} y_{ikjl} = 0 \\ \qquad\qquad (\forall i, \forall j \in N, \forall l \in H, i < j). \end{cases}$$

Then we have a reformulation of (HLP) as a mixed integer programming problem [2];

(MIP)

$$\begin{array}{ll} \text{min.} & \sum_{(i,j)\in A} w_{ij}(\sum_{k\in H} c_{ik}x_{ik} \\ & \quad + \sum_{k\in H}\sum_{l\in H} c_{kl}y_{ikjl} + \sum_{l\in H} c_{lj}x_{jl}) \\ \text{s. t.} & (x,y) \in \mathcal{A},\ y \geq 0, \\ & x_{ik} \in \{0,1\} \quad (\forall (i,k) \in N \times H). \end{array}$$

3. The polytope $\mathrm{HLP}_{n,p}$

Let G be a graph with vertex set $V := N \times H$ and edge set $E := \{\{(i,k),(j,l)\} \in \binom{V}{2} \mid i \neq j\}$. Then (MIP) turns into a problem to find a n-clique with minimal node- and edge- weight by setting a_{ik} for each node $(i,k) \in V$ and b_{ikjl} for each edge $\{(i,k),(j,l)\} \in E$, where $a_{ik} = \sum_{j\in N}(w_{ij}c_{ik} + w_{ji}c_{ki}) + \sum_{l\in H}(w_{il}(c_{ik}+c_{kl}) + w_{li}(c_{ki}+c_{lk}))$, $b_{ikjl} = (w_{ij}+w_{ji})c_{kl}$. Note that x_{ik} and y_{ikjl} correspond to each node and each edge of G respectively. And each feasible solution of (MIP) corresponds to an n-clique of G.

We introduce a polytope $\mathrm{HLP}_{n,p}$ as a convex hull of feasible solutions as follows:

$$\mathrm{HLP}_{n,p} = \mathrm{conv}\left\{ (x,y) \,\middle|\, \begin{array}{ll} x(\mathrm{row}_i) = 1 & (\forall i \in N), \\ x_v \in \{0,1\}, & (\forall v \in V), \\ y_e = x_u x_v & (\forall e \in E). \end{array} \right\},$$

where $x(T) = \sum_{v \in T} x_v$ $(T \subseteq V)$ and $\mathrm{row}_i = \{(i,k) \mid k \in H\}$.

To investigate the structure of $\mathrm{HLP}_{n,p}$, we apply the *star transformation* which was introduced to the quadratic assignment polytope [3]. This transformation induces a projection of $\mathrm{HLP}_{n,p}$ onto an affine subspace

$$\mathcal{U} = \left\{ (x,y) \,\middle|\, \begin{array}{ll} x_v = 0 & (\forall v \in V \setminus V^\star), \\ y_e = 0 & (\forall e \in E \setminus E^\star) \end{array} \right\},$$

where $H^\star = H \setminus \{p\}$, $V^\star = \{(i,k) \mid i \in N, k \in H^\star\}$, $E^\star = \{\{(i,k),(j,l)\} \mid i,j \in N, k,l \in H^\star, i < j\}$. Define the polytope $\mathrm{HLP}^\star_{n,p}$ as the image of $\mathrm{HLP}_{n,p}$. We can see that $\mathrm{HLP}^\star_{n,p}$ is a full dimensional polytope in $\mathrm{R}^{V^\star} \times \mathrm{R}^{E^\star}$ and is more tractable.

The following properties, concerning the linear constraints of (MIP), are shown by pulling back from $\mathrm{HLP}^\star_{n,p}$.

Theorem 1 *$\mathcal{A}$ is the affine hull of* $\mathrm{HLP}_{n,p}$.

Theorem 2 *The inequality $y_e \geq 0$ $(\forall e \in E)$ defines a facet of* $\mathrm{HLP}_{n,p}$.

4. New facet defining inequalities

Padberg introduced the *Boolean quadric polytope* BQP_n and showed two classes of valid inequalities called *clique-inequalities* and *cut-inequalities* [1]. Since $\mathrm{HLP}_{n,p}$ is a face of BQP_{np}, it is easy to see that the following lemma holds.

Lemma 3 $\forall \beta \in \mathrm{Z}, \forall S, \forall T \subset V$, *the clique-inequality*

$$(\beta - 1)x(T) - y(E(T)) \leq \frac{\beta(\beta - 1)}{2},$$

and the cut-inequality

$$-\beta x(S) + (\beta - 1)x(T) - y(E(S)) - y(E(T)) + y(E(S:T)) \leq \tfrac{\beta(\beta-1)}{2}$$

are valid for $\mathrm{HLP}_{n,p}$.

We found some new facet defining inequalities for $\mathrm{HLP}_{n,p}$. The following inequalities are among the simplest ones.

Theorem 4 *The clique-inequality with*

$$\beta = 2, \ T = \{u, v, w\} \ (u \prec v \prec w),$$

and the cut-inequality with

$$\beta = 1, \ S = \{u\}, \ T = \{v, w\} \quad (v \prec w, u \notin \mathrm{row}(v) \cup \mathrm{row}(w))$$

define facets of $\mathrm{HLP}_{n,p}$, *where* $(i,k) \prec (j,l) \overset{\text{def}}{\Leftrightarrow} i < j$ *and* $\mathrm{row}((i,k)) := \{(i,l) \mid l \in H\}$.

5. Computational experiments

We applied our approach to benchmark test problems. Computational experiments were performed for instances with 4 hub nodes. We found an integer optimal solution for every instance by adding the new facet defining inequalities as cutting planes.

References

[1] M. Padberg, The Boolean quadric polytope : some characteristics, facets and relatives, *Mathematical Programming*, 45 (1989), 139–172.

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[3] M. Jünger and V. Kaibel, The QAP-polytope and the star transformation, *Discrete Applied Mathematics*, 111 (2000), 283–306.