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Extending the Assignment Problem

東京農工大学 * 榊原 静 SAKAKIBARA Shizu
02103250 東京農工大学 嘉村 友作 KAMURA Yuusaku
01401940 東京農工大学 中森 眞理雄 NAKAMORI Mario

1 Introduction

Suppose you are an employer and you have n employees for n jobs. Each employee can do only one job, and each job is to be done by only one employee, so you must assign a job to each employee. Let the *cost* for the ith employee to do the jth job be denoted by c_{ij}. Assuming that it is the total cost you want to minimize, then we can state the assignment problem as,

P: Minimize

$$z = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, \ldots, n),$$
$$\sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, \ldots, n),$$
$$x_{ij} \geq 0 \quad (i = 1, \ldots, n;\ j = 1, \ldots, n). \qquad \square$$

The variable x_{ij} takes value 1 if the jth job is assigned to the ith employee, and 0 otherwise. It is well known that the optimal solution of P is integer valued, i.e., 0 or 1.

It is also known that the assignment problem is a special case of the minimum cost flow problem in a network [3], and efficient algorithms have been proposed, e.g., [1], [4], etc.

In this paper, we extend the assignment problem such that the cost c_{ij} is 2 dimensional, i.e., the cost for the ith employee to do the jth job is evaluated as c_{ij} from one point of view and as c'_{ij} from another point of view. Thus we have two total costs, $\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij}$ and $\sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}x_{ij}$, so the problem is:

P$'$: Minimize $\quad z$

subject to

$$\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij} \leq z,$$
$$\sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}x_{ij} \leq z,$$
$$\sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, \ldots, n),$$
$$\sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, \ldots, n),$$
$$x_{ij} = 0 \text{ or } 1 \quad (i = 1, \ldots, n;\ j = 1, \ldots, n). \qquad \square$$

The integral constraint in P$'$ is essential, because the unimodularity of the constraints does not hold.

Our goal is to propose an algorithm for problems P$'$. Rather than solving P$'$ exactly, which is NP-hard, we find a solution that is better than the two known subproblem solutions. The algorithm is also shown to be of polynomial time complexity.

2 Parametric Assignment

Since problem P$'$ is difficult to solve, we consider a modified version of P$'$, i.e., a parametric assignment problem:

Q$'$: Minimize

$$z = t\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}x_{ij} + (1-t)\sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}x_{ij}(t)$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, \ldots, n),$$
$$\sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, \ldots, n),$$
$$x_{ij} \geq 0 \quad (i = 1, \ldots, n;\ j = 1, \ldots, n) \qquad \square$$

for a given t $(0 \leq t \leq 1)$.

If t is fixed, problem Q$'$ is an ordinary assignment problem.

Let us denote the optimal solution of P$'$ by $\hat{x}_{ij}(t)$ and the value of the objective function by $F(t)$. Note that $\hat{x}_{ij}(1)$ and $\hat{x}_{ij}(0)$ are the optimal solutions of the ordinary assignment problem with costs c_{ij} and c'_{ij}. Noting that there are only a finite number of distinct $\hat{x}_{ij}(t)$'s (at most $n!$ in total), we have

Lemma 1 $F(t)$ is piecewise linear.
Also, we can show

Lemma 2 $F(t)$ is concave, i.e.,

$$F(\lambda t_1 + (1-\lambda)t_2) \geq \lambda F(t_1) + (1-\lambda)F(t_2)$$

for any λ $(0 \leq \lambda \leq 1)$, where $0 \leq t_1 \leq 1$ and $0 \leq t_2 \leq 1$.

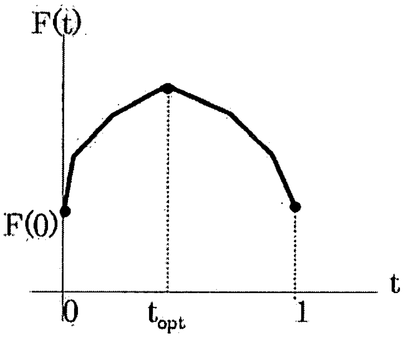

Figure 1: F(t)

Let the maximum value of $F(t)$ $(0 \leq t \leq 1)$ be $F(t_{\text{opt}})$ $(0 \leq t_{\text{opt}} \leq 1)$. It is not difficult to show

Theorem 1

$$\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\hat{x}_{ij}(t_{\text{opt}}) \leq \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\hat{x}_{ij}(0),$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}\hat{x}_{ij}(t_{\text{opt}}) \leq \sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}\hat{x}_{ij}(1).$$

Proof

Since $\hat{x}_{ij}(t_{\text{opt}})$ is the optimal solution for the assignment problem with cost $t_{\text{opt}}c_{ij} + (1-t_{\text{opt}})c'_{ij}$,

$$F(t_{\text{opt}}) \leq \sum_{j=1}^{n}\sum_{i=1}^{n}\{t_{\text{opt}}c_{ij} + (1-t_{\text{opt}})c'_{ij}\}\hat{x}_{ij}(1)$$

However,

$$\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\hat{x}_{ij}(1) \leq \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\hat{x}_{ij}(t_{\text{opt}}).$$

Therefore

$$\sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}\hat{x}_{ij}(t_{\text{opt}}) \leq \sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}\hat{x}_{ij}(1).$$

Similarly, we have

$$\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\hat{x}_{ij}(t_{\text{opt}}) \leq \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\hat{x}_{ij}(0).$$

Q.E.D.

Theorem 1 shows that $\hat{x}_{ij}(t_{\text{opt}})$ is a "better" solution than $\hat{x}_{ij}(0)$ or $\hat{x}_{ij}(1)$ for P2.

Finally, we show how to obtain t_{opt}. Since, the function $F(t)$ is a piecewise linear and concave function, it is clear that

$$\frac{d}{dt}F(t) > 0 \qquad \text{when} \qquad t < t_{\text{opt}}$$

and

$$\frac{d}{dt}F(t) < 0 \qquad \text{when} \qquad t > t_{\text{opt}}\,.$$

Note that

$$\frac{d}{dt}F(t) = \sum_{j=1}^{n}\sum_{i=1}^{n}(c_{ij} - c'_{ij})\hat{x}_{ij}(t),$$

where $\hat{x}_{ij}(t)$ is the optimal solution of P$'$. It is easy to obtain a t_{opt} by binary search in time $O(n^3L)$, where L is number of bits in the computer "word."

3 Conclusions

We have extended the classical assignment problem i.e., vector cost assignment. We considered a parametric assignment problem whose optimal solution is better than the known ones. Detailed analysis and the development of more efficient algorithms of the parametric assignment problem are left for further research.

References

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