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Multicoloring Unit Disk Graphs on Triangular Lattice Points

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1 Introduction

Given a pair of non-negative integers m and n, $P(m,n)$ denotes the subset of 2-dimensional integer triangular lattice points defined by $P(m,n) \stackrel{\text{def.}}{=} \{(xe_1+ye_2) \mid x \in \{0,\dots,m-1\},\ y \in \{0,\dots,n-1\}\}$ where $e_1 \stackrel{\text{def.}}{=} (1,0)$, $e_2 \stackrel{\text{def.}}{=} (1/2, \sqrt{3}/2)$. Given a finite set of 2-dimensional points $P \subseteq \mathrm{R}^2$ and a positive real d, a *unit disk graph*, denoted by (P,d), is an undirected graph with vertex set P such that two vertices are adjacent if and only if the Euclidean distance between the pair is less than or equal to d. We denote the unit disk graph $(P(m,n),d)$ by $T_{m,n}(d)$.

Given an undirected graph H and a non-negative integer vertex weight $\boldsymbol{w}'$ of H, a *multicoloring* of $(H,\boldsymbol{w}')$ is an assignment of colors to vertices of H such that each vertex v admits $w'(v)$ colors and every adjacent pair of two vertices does not share a common color. A *multicoloring problem* on $(H,\boldsymbol{w}')$ finds a multicoloring of $(H,\boldsymbol{w}')$ which minimizes the required number of colors.

The multicoloring problem has been studied in several context. When a given graph is the triangular lattice graph $T_{m,n}(1)$, the problem is related to the radio channel (frequency) assignment problem. McDiarmid and Reed [3] showed that the multicoloring problem on triangular lattice graphs is NP-hard. Some authors [3, 5] independently gave (4/3)-approximation algorithms for this problem. For coloring (general) unit disk graphs, there exists a 3-approximation algorithm [2, 6]. Here we note that the approximation ratio of our algorithm is less than $1+2/\sqrt{3} < 2.155$ for any $d \geq 1$.

2 Well-Solvable Cases and Perfectness

An undirected graph G is *perfect* if for each induced subgraph H of G, the chromatic number of H, denoted by $\chi(H)$, is equal to its clique number $\omega(H)$. The following theorem is a main result of this paper.

Theorem 1 [4] *When $n \geq 1$ and $d \geq 1$, we have the following; [$\forall m \in \mathrm{Z}_+$, $T_{m,n}(d)$ is perfect] if and only if $d \geq \sqrt{n^2-3n+3}$.*

Table 1 shows the perfectness and imperfectness of $T_{m,n}(d)$ for small n and d.

Table 1: Perfectness and imperfectness

n \ d	$1\cdots$	$\sqrt{7}\cdots$	$\sqrt{13}\cdots$	$\sqrt{21}\cdots$
1	Perfect	Perfect	Perfect	Perfect
2	Perfect	Perfect	Perfect	Perfect
3	Perfect	Perfect	Perfect	Perfect
4	Imperfect	Perfect	Perfect	Perfect
5	Imperfect	Imperfect	Perfect	Perfect
6	Imperfect	Imperfect	Imperfect	Perfect
$\vdots$	Imperfect	Imperfect	Imperfect	Imperfect

An undirected graph which is transitively orientable is called *comparability graph*. The complement of a comparability graph is called *co-comparability graph*. It is well-known that every co-comparability graph is perfect.

Lemma 1 *Let $d > 1$ be a real number. Then, $T_{m,n}(d)$ is a co-comparability graph, if and only if $n \leq \frac{3+\sqrt{4d^2-3}}{2}$.*

The following lemma deals with the special case that $n=3$, $d=1$.

Lemma 2 *For $\forall m \in \mathrm{Z}_+$ and $1 \leq \forall d < \sqrt{3}$, the graph $T_{m,3}(d)$ is perfect.*

Note that though the graph $T_{m,3}(1)$ is perfect, the graph $T_{m,3}(1)$ is not co-comparability graph.

From the above, the perfectness of a graph satisfying the conditions of Theorem 1 is clear. In the following, we discuss the inverse implication. We say that an undirected graph G has an *odd-hole*, if G contains an induced subgraph isomorphic to an odd cycle whose length is greater than or equal to 5. It is obvious that if a graph has an odd-hole, the graph is not perfect.

Lemma 3 *For $\forall n \geq 4$, if $1 \leq d < \sqrt{n^2-3n+3}$, then $\exists m \in \mathrm{Z}_+$, $T_{m,n}(d)$ has an odd-hole.*

Lemma 3 shows the imperfectness of every graph which violates a condition of Theorem 1.

Given an undirected graph $G=(V,E)$ and vertex weight vector $\boldsymbol{w} \in \mathrm{Z}_+^V$, the *multicoloring number*

$\chi(G, \boldsymbol{w})$ is the least number of colors required in a multicoloring of $(G, \boldsymbol{w})$. The *weighted clique number* $\omega(G, \boldsymbol{w})$ is the weight of a maximum weight clique in $(G, \boldsymbol{w})$. It is clear that $\chi(G, \boldsymbol{w}) \geq \omega(G, \boldsymbol{w})$.

Then we have the following.

Theorem 2 [4] *When $n \geq 1$, the following property holds;* [$\forall m \in \mathrm{Z}_+$ *and* $\forall \boldsymbol{w} \in \mathrm{Z}_+^{P(m,n)}$,
$\chi(T_{m,n}(d), \boldsymbol{w}) = \omega(T_{m,n}(d), \boldsymbol{w})$]
if and only if $d \geq \sqrt{n^2 - 3n + 3}$.

Assume that we have a co-comparability graph G and related digraph H which gives a transitive orientation of the complement of G. Then each independent set of G corresponds to a chain (directed path) of H. The multicoloring problem on G is essentially equivalent to the minimum size chain cover problem on H. Every clique of G corresponds to an anti-chain of H. Thus the equality $\omega(G) = \chi(G)$ is obtained from Dilworth's decomposition theorem. It is well-known that the minimum size chain cover problem on an acyclic graph is solvable in polynomial time by using an algorithm for minimum-cost circulation flow problem.

In case that a given graph is $(T_{m,3}(1), \boldsymbol{w})$, we proposed a strongly polynomial time algorithm for muticoloring $(T_{m,3}(1), \boldsymbol{w})$ (see [4]).

3 Approximation Algorithm

When $d = 1$, McDiarmid and Reed [3] proposed an approximation algorithm for $(T_{m,n}(1), \boldsymbol{w})$, which finds a multicoloring with at most $(4/3)\omega(T_{m,n}(1), \boldsymbol{w}) + 1/3$ colors.

Theorem 3 [4] *When $d > 1$, there exists a polynomial time algorithm for multicoloring $(T_{m,n}(d), \boldsymbol{w})$ such that the number of required colors is bounded by*

$$\left(1 + \frac{\lfloor \frac{2}{\sqrt{3}} d \rfloor}{\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor}\right) \omega(T_{m,n}(d), \boldsymbol{w}) + \left(\left\lfloor \frac{3+\sqrt{4d^2-3}}{2} \right\rfloor - 1\right) \chi(T_{m,n}(d)).$$

Proof: We describe an outline of the algorithm. For simplicity, we define $K_1 \stackrel{\text{def.}}{=} \lfloor \frac{3+\sqrt{4d^2-3}}{2} \rfloor$ and $K_2 \stackrel{\text{def.}}{=} \lfloor \frac{3+\sqrt{4d^2-3}}{2} \rfloor + \lfloor \frac{2}{\sqrt{3}} d \rfloor$.

First, we construct K_2 vertex weights $\boldsymbol{w}'_k$ for $k \in \{0, 1, \ldots, K_2 - 1\}$ by setting

$$w'_k(x, y) = \begin{cases} 0, & y \in \{k, \ldots, k + \lfloor \frac{2}{\sqrt{3}} d \rfloor - 1\} \pmod{K_2}, \\ K_1 \left\lfloor \frac{w(x,y)}{K_1} \right\rfloor, & \text{otherwise.} \end{cases}$$

Next, we exactly solve K_2 multicoloring problems defined by K_2 pairs $(T_{m,n}(d), \boldsymbol{w}'_k)$, $k \in \{0, 1, \ldots, K_2 - 1\}$ and obtain K_2 multicolorings. We can solve each problem exactly in polynomial time, since every connected component of the graph induced by the set of vertices with positive weight is a perfect graph discussed in the previous section. Thus $\chi(T_{m,n}(d), \boldsymbol{w}'_k) = \omega(T_{m,n}(d), \boldsymbol{w}'_k)$ for any $k \in \{0, 1, \ldots, K_2 - 1\}$. Put $\boldsymbol{w}'' = \boldsymbol{w} - \sum_{k=0}^{K_2-1} \boldsymbol{w}'_k$. Then each element of $\boldsymbol{w}''$ is less than or equal to $K_1 - 1$. Thus we can find a multicoloring of $(T_{m,n}(d), \boldsymbol{w}'')$ from the direct sum of $K_1 - 1$ trivial colorings of $T_{m,n}(d)$. The obtained multicoloring uses at most $(K_1 - 1)\chi(T_{m,n}(d))$colors. Lastly, we output the direct sum of $K_2 + 1$ multicolorings obtained above. The definition of the weight vector $\boldsymbol{w}'_k$ implies that $\forall k \in \{0, 1, \ldots, K_2 - 1\}$, $K_1 \omega(T_{m,n}(d), \boldsymbol{w}'_k) \leq \omega(T_{m,n}(d), \boldsymbol{w})$. Thus, the obtained multicoloring uses at most
$(K_2/K_1)\omega(T_{m,n}(d), \boldsymbol{w}) + (K_1 - 1)\chi(T_{m,n}(d))$ colors. ∎

We have also shown the following hardness result.

Theorem 4 [4] *Let d be a constant rational number. Given a pair $(T_{m,n}(d), \boldsymbol{w})$, it is NP-complete to determine whether $(T_{m,n}(d), \boldsymbol{w})$ is multicolorable with strictly less than $(4/3)\omega(T_{m,n}(d), \boldsymbol{w})$ colors or not.*

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