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On the Value – Volatility Relationship in a Real Options Model

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1 Introduction

In the analytical real options approach, the most important proposition that the value of the investment opportunity increases as the volatility increases has been proved by assuming the convexity of the drift of the stochastic differential equation defined as the state variable. This paper demonstrates numerically that the convexity of the drift is not necessary for that proposition in the real options approach.

2 Some Preliminaries

Consider a firm having the possibility to make an irreversible investment that increases his profits. We assume that the firm is risk neutral. We denote a state variable that the revenue for the investment depends upon by $(X_t)_{t\in\mathbb{R}_+}$ that is defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$.

It is assumed that the revenue process $(X_t)_{t\in\mathbb{R}_+}$ evolves according to

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}z_t, \tag{1}$$

where $X_0 =: x \in \mathbb{R}_{++}$. Both the infinitesimal drift $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ and the infinitesimal diffusion coefficient $\sigma : \mathbb{R}_+ \to \mathbb{R}_+\backslash\{0\}$ are assumed to be Lipschitz continuous. Here, $(z_t)_{t\in\mathbb{R}_+}$ denotes a one-dimensional $\mathbb{P}$–standard Brownian motion. We will also assume that both the lower boundary 0 and the upper boundary ∞ are natural for the revenue process $(X)_{t\in\mathbb{R}_+}$.

Suppose that the current time is $t \in \mathbb{R}_+$, and let τ be the stopping time at which the firm adopts the investment opportunity after time t. We denote the set of admissible strategies at time t by $\mathcal{T}_t$. The risk neutral discount factor is constant and equals $r \in \mathbb{R}_{++}$. The value function of the investment opportunity is given by

$$C(x) := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}^x\left[\mathrm{e}^{-r\tau}\{X_\tau - I\}\,|\mathcal{F}_t\right], \tag{2}$$

where $X_t =: x \in \mathbb{R}_{++}$. $\mathbb{E}^x[\cdot|\mathcal{F}_t]$ stands for the conditional expectation operator evaluated at the initial state $x = X_t$ with respect to the risk neutral measure $\mathbb{P}$. And $I \in \mathbb{R}_{++}$ stands for the sunk cost.

After some algebra, the value in the continuation region satisfies the following ordinary differential equation (ODE):

$$\frac{1}{2}\sigma^2(x)C''(x) + \mu(x)C'(x) - rC(x) = 0, \tag{3}$$

where $x < x^*$. x^* stands for the optimal threshold. The corresponding boundary conditions turn out to be

$$C(0) = 0, \quad C(x^*) = x^* - I, \quad C'(x^*) = 1. \tag{4}$$

According to Theorem 2 or 3 in Alvarez and Stenbacka (2001), we state an important lemma.

Lemma 2.1 *Assume that the drift term $\mu(x)$ on Equation (1) is convex in x. Then, the value function $C(x)$ is increasing and convex in x. Moreover, the value $C(x)$ increases as the volatility σ increases.*

3 Main Results

In this section, we assume that the revenue process, i.e., the state variable, follows a non–linear SDE with concave drift. The evolution of the revenue process is defined as:

$$\mathrm{d}X_t = \kappa X_t(m - X_t)\,\mathrm{d}t + \sigma X_t\,\mathrm{d}z_t, \tag{5}$$

where κ, m, and σ are some constants (i.e., $\kappa, m, \sigma \in \mathbb{R}_+$). Equation (5) is often used as a model for the growth of a population size in a stochastic, crowded environment. It should be noted that the drift $\mu(x)$ of (5) is not convex, but concave in x. We can obtain the following lemma.

Lemma 3.1 *The (strong) solution of Equation (5) is given by:*

$$X_T = \frac{e^{\left(\kappa m - \frac{1}{2}\sigma^2\right)T + \sigma z_T}}{x^{-1} + \int_0^T \kappa e^{\left(\kappa m - \frac{1}{2}\sigma^2\right)t + \sigma z_t} dt}. \quad (6)$$

Note that the explicit solution is not necessarily obtained in many nonlinear cases.

Similar to derivation of (3), the Bellman equation that the value function must satisfy in the continuation region is given by

$$\frac{1}{2}\sigma^2 x^2 C''(x) + \kappa(m-x)xC'(x) - rC(x) = 0, \quad (7)$$

where $x < x^*$. Also the value function must satisfy the boundary conditions (4) as before.

We use a numerical method to calculate the value of the firm and optimal threshold.[1] Figure 1 depicts the value $C(x)$ with respect to the initial state x.[2] Figure 1 shows that the value $C(x)$ is no longer convex for all x; it is concave for small value of x.

Proposition 3.1 *We assume that the state variable follows Equation (5). Then, the value $C(x)$ is not necessarily convex in x.*

According to analytical results on the real options model, we cannot prove the most important proposition in our setting. This is because the drift of (5) is not convex in x. In what follows, we investigate whether the monotonicity of the volatility on the value is obtained or not by using numerical methods when the value is not convex for all x.

Figure 2 depicts the value for several choices of volatility σ.

Proposition 3.2 *The value $C(x)$ increases as the volatility σ increases even when the state variable follows the nonlinear SDE (5), while it is not convex for all x.*

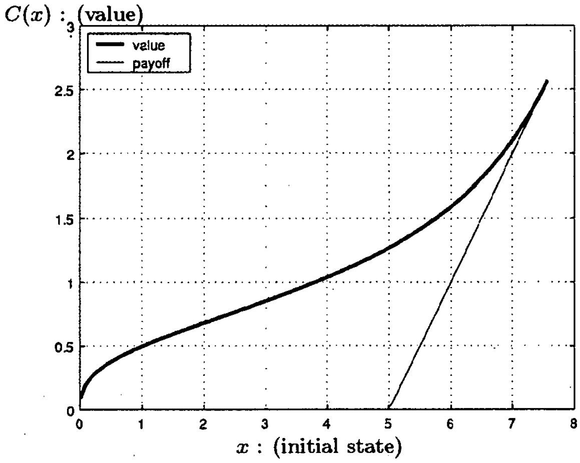

Figure 1. Value of the firm

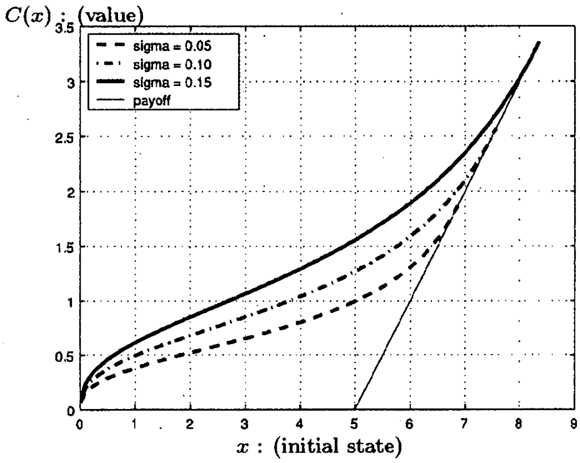

Figure 2. Impact of volatility σ

In this paper, we demonstrate that the monotonicity of volatility on the value is quite robust.

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References

[1] Alveretz and Stenbacka, (2001), "Adoption of Uncertain Multi-stage Technological Project", Journal of Mathematical Economics, 63 pp211-234.

[1] We set the basic parameters as $I = 5$, $m = 5$, $r = 0.05$, $\sigma = 1.0$, and $\kappa = 0.02$.

[2] With these parameters, we find that the optimal threshold x^* is calculated as 7.5692.