Risk-Sensitive Preventive Maintenance Policies

Bibhas C. GIRI and Tadashi DOHI (01307065)
Department of Information Engineering, Hiroshima University

1. Introduction

Preventive maintenance policies are generally determined by minimizing the expected cost per unit time in the steady state. However, maintenance managers are often not satisfied with only the estimation of the expected cost. They require some additional information regarding the variability of the cost to evaluate the risk involved in industrial maintenance. In some cases, they may prefer to accept some degree of increased expected cost in order to reduce the variability of the cost. In a recent paper, Chen and Jin (2003) introduced the concept of long-run variance of the cost to represent management risk in preventive maintenance policies. Unfortunately, their analysis involves serious mistakes. In this paper, we offer the exact formulation of the long-run variance of the cost for the age replacement and block replacement policies. In numerical examples, we study the impacts of risk-sensitive optimality criterion on the managerial decision process.

2. Nomenclature

a (≥ 0) : preventive replacement interval; μ : risk-sensitive factor; $C(t;a)$: accumulated cost during the time interval $(0,t]$ when a system begins to operate at time 0; $M(t;a)$: $E[C(t;a)]$; $V(t;a)$: $\mathrm{Var}[C(t;a)]$; $F(t)$: failure time distribution; $\overline{F}(t)$: survivor function of $F(t)$; c_1, c_2 ($c_1 > c_2 > 0$) : costs of a failure replacement and a preventive replacement, respectively for the age replacement policy; c_r, c_m (> 0) : costs of a preventive replacement and a minimal repair, respectively for the block replacement (with minimal repair) policy; $\Phi(a)$ ($\Theta(a)$) : long-run average cost for the age replacement (block replacement with minimal repair) policy; $\Psi(a)$ ($\Omega(a)$) : long-run variance of the cost for the age replacement (block replacement with minimal repair) policy. When there is no confusion, we abbreviate $C(t;a), M(t;a), V(t;a)$, as $C(t), M(t), V(t)$, respectively.

3. Risk-sensitive age replacement policy

It is well known in the age replacement policy that if the system does not fail until a prescribed time a then it is replaced by a new one preventively. Otherwise, it is replaced at the failure age. Under this policy, the expected cost per unit time in the steady state is given by (Barlow and Proschan, 1965)

$$\Phi(a) = \frac{c_1 F(a) + c_2 \overline{F}(a)}{\int_0^a \overline{F}(t)dt}. \quad (1)$$

The variance of the accumulated cost $V(t)$ is given by

$$\begin{aligned} V(t) &= E\left[\{C(t) - M(t)\}^2\right] \\ &= E\left[\{C(t) - \Phi t\}^2\right] - [M(t) - \Phi t]^2. \quad (2) \end{aligned}$$

Proposition 1 [Kawai and Tanaka (1987)]: $\lim_{t\to\infty}[M(t) - \Phi t]$ exists and is equal to

$$\left[c_1 \int_0^a t dF(t) + c_2 a\, \overline{F}(a) + \Phi \int_0^a t\, \overline{F}(t)dt\right] / \int_0^a \overline{F}(t)dt.$$

Proposition 2 [Kawai and Tanaka (1987)]: Using Proposition 1, the long-run variance of the accumulated cost is given by

$$\begin{aligned} \Psi(a) &= \lim_{t\to\infty} \frac{V(t)}{t} \\ &= \frac{\int_0^a (c_1 - \Phi t)^2 dF(t) + (c_2 - \Phi a)^2\, \overline{F}(a)}{\int_0^a \overline{F}(t)dt}. \quad (3) \end{aligned}$$

Hence, following Chen and Jin (2003), the risk-sensitive age replacement problem can be formulated as

$$\min_{a\in[0,\infty)} \Pi(a) = [\Phi(a)]^2 + \mu\Psi(a). \quad (4)$$

The decision maker is said to be risk averter if $\mu > 0$, risk seeking if $\mu < 0$ and risk neutral if $\mu = 0$. Since the decision makers are usually risk averse, we assume that $\mu \geq 0$.

Remark: Chen and Jin (2003) defined the long-run variance of the cost $V_1(a)$ for the age replacement policy with replacement interval a as

$$V_1(a) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} [C_t - \Phi(a)]^2,$$

where C_t is the cost spent at time t. Clearly, $V_1(a)$ can not represent the long-run variance of the cost under the age replacement policy. Hence their cost-variability-sensitive analysis is incorrect.

4. Risk-sensitive block replacement policy with minimal repair

Suppose that the preventive replacement is executed periodically at a pre-specified time ka ($k = 1, 2, 3, \cdots$) with a repair cost c_r (> 0). If the unit fails within a periodic replacement interval then minimal repair is performed with a cost c_m (> 0) and the failure rate is not disturbed by each repair. Let $N(t;a)$ denote the number of minimal repairs up to time t. Then, the stochastic process $\{N(t;a), t \geq 0\}$ is governed by a non-homogeneous Poisson process (NHPP) with an intensity function $r(t)$ and a mean value function $\Lambda(t)$, *i.e.*,

$$\Lambda(t) = \int_0^t r(x)dx,$$

where $r(t) = f(t)/\overline{F}(t)$ is the failure rate (or hazard rate) of the failure time distribution. Following Barlow and Hunter (1960), the expected cost per unit time in the steady state is given by

$$\Theta(a) = \frac{c_m\Lambda(a) + c_r}{a}. \quad (5)$$

Let K be a random variable denoting the total cost incurred during a preventive replacement cycle. Therefore, K takes the values $c_r, c_r+c_m, c_r+2c_m, c_r+3c_m, \cdots, c_r+jc_m, \cdots$ for $N(t;a) = 0, 1, 2, 3, \cdots j, \cdots$, respectively. Then we have,

$$E[K^2] = \sum_{n=0}^{\infty} (c_r + nc_m)^2 \; \frac{e^{-\Lambda(a)}\{\Lambda(a)\}^n}{n!}$$
$$= c_r^2 + \left(2c_rc_m + c_m^2\right)\Lambda(a) + c_m^2\{\Lambda(a)\}^2. \quad (6)$$

Therefore, $Var[K] = E[K^2] - \{E[K]\}^2 = c_m^2\Lambda(a)$. Hence, by applying Wald's first and second moment identities and renewal theory convergence theorems, the asymptotic variance per unit time can be obtained as

$$\Omega(a) = \lim_{t\to\infty} \frac{Var[C(t;a)]}{t} = \frac{Var[K]}{a} = \frac{c_m^2\Lambda(a)}{a}. \quad (7)$$

Therefore, the optimization problem for the risk-sensitive block replacement with minimal repair is

$$\min_{a\in[0,\infty)} S(a) = \left[\left\{\frac{c_m\Lambda(a) + c_r}{a}\right\}^2 + \mu\,\frac{c_m^2\Lambda(a)}{a}\right]. \quad (8)$$

Define the numerator of the derivative of $S(a)$ with respect to a as $Z(a)$.

Theorem: (1) Suppose that the $F(t)$ has strictly increasing failure rate (IFR) property.

(i) If $Z(\infty) > 0$ then there exists a unique optimal block replacement (with minimal repair) interval a^* ($0 < a^* < \infty$) satisfying $Z(a^*) = 0$ which minimizes $S(a)$. The corresponding time average expected cost and variance in the steady state are $\Theta(a^*) = [c_m\Lambda(a^*) + c_r]/a^*$ and $\Omega(a^*) = [c_m^2\Lambda(a^*)]/a^*$, respectively.

(ii) If $Z(\infty) \leq 0$ then $a^* \to \infty$ and the corresponding time average expected cost and variance in the steady state are $\Theta(\infty) = c_m r(\infty)$ and $\Omega(\infty) = c_m^2 r(\infty)$, respectively.

(2) Suppose that $F(t)$ has the strictly decreasing failure rate (DFR) property, *i.e.*, the failure rate is decreasing. Then the optimal block replacement (with minimal repair) interval is $a \to \infty$.

6. Numerical example

Suppose that the failure time follows the gamma distribution whose probability density function is $f(t) = \lambda e^{-\lambda t}(\lambda t)^{n-1}/\Gamma(n)$, $t \geq 0$; $\lambda > 0$; $n > 0$.

Table 1 Risk-sensitive age replacement policy ($\lambda = 0.2$, $n = 3$, $c_1 = 2000$, $c_2 = 400$)

μ	a^*	$\Phi(a^*)$	$\Psi(a^*)$	$\Pi(a^*)$
0.00	7.56217	100.103	75161.8	10020.6
0.02	6.95934	100.397	69130.4	11462.1
0.05	6.26719	101.661	61695.5	13419.8
0.10	5.49360	104.811	52821.0	16267.5
0.15	4.98881	108.316	46775.4	18748.6
0.20	4.63019	111.774	42394.9	20972.5
0.50	3.58713	129.003	29535.9	31409.8
1.00	2.94250	148.792	21806.7	43945.9
5.00	1.87335	219.575	10317.3	99799.4

Table 2 Risk-sensitive block replacement policy ($\lambda = 0.5$, $n = 3$, $c_r = 1000$, $c_m = 200$)

μ	a^*	$\Theta(a^*)$	$\Omega(a^*)$	$S(a^*)$
0.00	89.5767	95.6342	16894.10	9145.89
0.02	84.8962	95.6405	16772.3	9482.55
0.05	78.6279	95.6736	16591.1	9983.0
0.10	69.8386	95.7898	16294.2	10805.1
0.15	62.6826	95.9782	16005.0	11612.5
0.20	56.7818	96.2334	15724.4	12405.7
0.50	36.2297	98.8379	14247.2	16892.5
1.00	23.3732	105.1840	12480.0	23543.8
2.00	15.0135	118.6190	10402.5	34875.6
5.00	8.9432	150.5670	7750.0	61420.5

The impacts of risk-sensitive optimality criterion on the age and block replacememt polices are shown in Tables 1 and 2, repectively. Note that the risk-sensitive preventive maintenance policy always reduces the cost variation significantly with a small increase in the average cost in the steady state.

References

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