A Conventional Scheme for Coping with Negative Output Data in DEA: A Slacks-based Measure (SBM) Approach *

01302170 National Graduate Institute for Policy Studies Kaoru Tone

1 Introduction

This paper proposes a new scheme for coping with negative output data, based on the slacks-based measure (SBM) proposed by Tone (2001). This scheme can definitely discern the above two DMUs.

In this paper, we propose a new scheme for coping with negative output data, based on the slacks-based measure (SBM) proposed by Tone (2001). This scheme can definitely discern the above two DMUs.

Throughout this paper, we deal with n DMUs with the input and output matrices $X = (x_{ij}) \in R^{m \times n}$ and $Y = (y_{ij}) \in R^{s \times n}$, where $m$ and $s$ are the numbers of input and output items, respectively. We assume that the observed input data set $X$ is positive (or at least non-negative). However, the observed output data set $Y$ is arbitrary, i.e., positive, zero or negative.

The production possibility set $P$ is defined as

$$P = \{(x, y) | x \geq X\lambda, y \leq Y\lambda, \lambda \geq 0\}, \quad (1)$$

where $\lambda$ is a nonnegative vector in $R^n$. (We can impose some constraints on $\lambda$, such as $\sum_{j=1}^n \lambda_j = 1$ (the variable returns-to-scale case), if it is needed to modify the production possibility set.)

*This research is supported by Grant-in-Aid for Scientific Research (C) Japan Society for the Promotion of Science.

2 The case by radial approaches

In this section we demonstrate that the traditional radial models cannot duly cope with negative output data.

The output-oriented CCR model for evaluating the relative efficiency of a DMU $(x_{io}, y_{io})$ is described by the linear program with variables $\eta$, $\lambda$, $s^-$, $s^+$ as below:

$$\begin{align*}
\eta^* &= \max \eta \\
\text{subject to}
\end{align*}$$

$$x_{io} = \sum_{j=1}^n x_{ij}\lambda_j + s^-_i \quad (i = 1, \ldots, m) \quad (3)$$

$$\eta_{iro} = \sum_{j=1}^n y_{rj}\lambda_j - s^+_r \quad (r = 1, \ldots, s) \quad (4)$$

$$\lambda_j \geq 0 \quad (\forall j), \quad s^-_i \geq 0 \quad (\forall i), \quad s^+_r \geq 0 \quad (\forall r),$$

where $s^-_i$ and $s^+_r$ designate slacks to input $i$ and output $r$, respectively.

Apparently, [CCR-O] has a feasible solution given by $\eta = 1$, $\lambda = 1$, $\lambda_j = 0 \quad (j \neq o)$, $s^- = 0$, $s^+ = 0$, and hence we have $\eta^* \geq 1$.

Let us assume that [CCR-O] has a finite optimum $\eta^* \quad (< \infty)$ and $y_o$ contains negative elements. Then, without losing generality, we can assume that $y_{1o} < 0$. The corresponding constraint in [CCR-O] is:

$$\eta_{1o} = y_{11}\lambda_1 + \cdots + y_{1n}\lambda_n - s^+_1.$$ 

This constraint cannot be a binding one to $\eta$, since $y_{1o} < 0$ and $s^+_1 \geq 0$, i.e., we can enlarge $\eta$ as large as possible by enlarging $s^+_1$ together. Therefore, we can delete this constraint from the above LP [CCR-O]. This leads to the statement that the magnitude of the negative output $y_{1o}$ is irrelevant to the efficiency score $\eta^*$ of the DMU $(x_{io}, y_{io})$, if $\eta^*$ is finite.

Thus, in order to overcome this difficulty, we need to translate the negative output to positive by adding a positive common number to all outputs in the same output item. However, this operation deforms the production possibility set, and
usually produces a wrong efficiency score, except for the input-oriented BCC case (see Ali and Seiford (1990)).

3 A scheme using the non-radial SBM approaches

The slacks-based measure (SBM) evaluates the relative efficiency of DMU \((x_o, y_o)\) by solving the following program with variables \(\lambda, s^-, s^+\).

\[
\begin{align*}
\text{[SBM]} & \quad \rho^* = \min \rho = \frac{1 - \frac{1}{m} \sum_{i=1}^{m} \frac{x_{io}}{y_{io}}}{1 + \frac{1}{s} \sum_{r=1}^{s} \frac{s_r}{y_{ro}}} \\
\text{subject to} & \quad x_o = X\lambda + s^- \\
& \quad y_o = Y\lambda - s^+ \\
& \quad \lambda \geq 0, \quad s^- \geq 0, \quad s^+ \geq 0.
\end{align*}
\]

The [SBM] encounters trouble with non-positive \(y_{ro}\), since the term \(s^+/y_{ro}\) is indefinite if \(y_{ro} = 0\) and negative if \(y_{ro} < 0\).

3.1 A new scheme for resolving the non-positive problem

In an effort to resolve this problem with the case \(y_{ro} \leq 0\), we modify the term \(s^+/y_{ro}\) in the objective function as follows.

First, we define \(y^M_r\) and \(y^m_r\) by

\[
\begin{align*}
y^M_r & = \max_{j=1,...,n} \{y_{rj}|y_{rj} > 0\} \\
y^m_r & = \min_{j=1,...,n} \{y_{rj}|y_{rj} > 0\}.
\end{align*}
\]

If the output \(r\) has no positive element, we define

\[
y^M_r = y^m_r = 1.
\]

Secondly, we replace the term \(s^+/y_{ro}\) in the objective function in the following way. (We notice that we never change the value \(y_{ro}\) in the constraint.)

[Case 1] If \(y^M_r > y^m_r\), we replace the term by

\[
s^+ = \frac{y^m_r(y^M_r - y^m_r)}{y^M_r - y_{ro}}.
\]

[Case 2] If \(y^M_r = y^m_r\), we replace the term by

\[
s^+ = \frac{(y^M_r)^2}{B(y^M_r - y_{ro})}
\]

where \(B\) is a large positive number, e.g., \(B=100\).

In any case, the denominator is positive and strictly less than \(y^M_r\). Furthermore, it is in reverse proportion to the distance \(y^M_r - y_{ro}\). Thus, this scheme takes into account the magnitude of the non-positive output. The score thus obtained is units invariant, i.e., it is independent of the units of measurement used.

In order to demonstrate the suitability of this scheme, we examine the case with two DMUs \((x_1, y_1)\) and \((x_2, y_2)\) that have the same inputs and outputs except for the first element of the outputs, i.e., \(x_1 = x_2\) and \(y_1 = y_{2r} \quad (r = 2, \ldots, s)\). As for the first element, we assume that \(y_1 > y_2\). Let the optimal objective values for \((x_1, y_1)\) and \((x_2, y_2)\) be \(\rho_1^*\) and \(\rho_2^*\), respectively. Then we have the theorem:

**Theorem 1** The optimal objective value \(\rho_1^*\) for DMU \((x_1, y_1)\) is greater than that \(\rho_2^*\) for DMU \((x_2, y_2)\).

3.2 The efficient status and projection

A DMU is efficient if and only if \(\rho^* = 1\), i.e., \(s^- = 0\) and \(s^{**} = 0\).

A projection to efficient frontiers is given by

\[
\begin{align*}
x_o & \leftarrow x_o - s^- \\
y_o & \leftarrow y_o + s^{**}.
\end{align*}
\]

4 An illustrative example

We exhibits a simple example with a single input and output.

<table>
<thead>
<tr>
<th>Table 1: A Sample Negative Data Set and Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>DMU</td>
</tr>
<tr>
<td>A</td>
</tr>
<tr>
<td>B</td>
</tr>
<tr>
<td>C</td>
</tr>
<tr>
<td>D</td>
</tr>
<tr>
<td>E</td>
</tr>
<tr>
<td>F</td>
</tr>
<tr>
<td>G</td>
</tr>
</tbody>
</table>

Reference

Charnes, Cooper, Rhodes (1978), EJOR 2, 429-444.
Tone (2001), EJOR 130, 498-509.