A Conventional Scheme for Coping with Negative Output Data in DEA: A Slacks-based Measure (SBM) Approach *

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1 Introduction

This paper proposes a new scheme for coping with negative output data in Data Envelopment Analysis (DEA). Since the inaugural paper by Charnes, Cooper and Rhodes (1978), most DEA models assume positive (or at least non-negative) data set for measuring relative efficiency of decision-making units (DMUs).

However, in several occasions, we are forced to deal with negative data in outputs. For example, if we choose operating profits as an output item, the observed value may vary from positive (profit) to negative (loss). Thus, we need a scheme for coping duly with negative data. The scheme should recognize a DMU with a small loss to be more efficient than a DMU with a large loss, if the remaining observed data of the two DMUs are same. As we see later, the traditional DEA models, e.g., the CCR and BCC models, cannot discriminate them properly, i.e., they give the same efficiency score to the two DMUs.

In this paper, we propose a new scheme for coping with negative output data, based on the slacks-based measure (SBM) proposed by Tone (2001). This scheme can definitely discern the above two DMUs.

Throughout this paper, we deal with n DMUs with the input and output matrices $X = (x_{ij}) \in R^{m\times n}$ and $Y = (y_{ij}) \in R^{s\times n}$, where m and s are the numbers of input and output items, respectively. We assume that the observed input data set X is positive (or at least non-negative). However, the observed output data set Y is arbitrary, i.e., positive, zero or negative.

The production possibility set P is defined as

$$P = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{x} \geq X\boldsymbol{\lambda}, \boldsymbol{y} \leq Y\boldsymbol{\lambda}, \boldsymbol{\lambda} \geq \boldsymbol{0}\}, \quad (1)$$

where $\boldsymbol{\lambda}$ is a nonnegative vector in R^n. (We can impose some constraints on $\boldsymbol{\lambda}$, such as $\sum_{j=1}^{n} \lambda_j = 1$ (the variable returns-to-scale case), if it is needed to modify the production possibility set.)

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2 The case by radial approaches

In this section we demonstrate that the traditional radial models cannot duly cope with negative output data.

The output-oriented CCR model for evaluating the relative efficiency of a DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is described by the linear program with variables η, $\boldsymbol{\lambda}$, $\boldsymbol{s}^-$, $\boldsymbol{s}^+$ as below:

$$[\text{CCR-O}] \quad \eta^* = \max \eta \quad (2)$$

subject to

$$x_{io} = \sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- \quad (i = 1, \ldots, m) \quad (3)$$

$$\eta y_{ro} = \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ \quad (r = 1, \ldots, s) \quad (4)$$

$$\lambda_j \geq 0 \ (\forall j),\ s_i^- \geq 0 \ (\forall i),\ s_r^+ \geq 0,\ (\forall r),$$

where s_i^- and s_r^+ designate slacks to input i and output r, respectively.

Apparently, [CCR-O] has a feasible solution given by $\eta = 1$, $\lambda_o = 1$, $\lambda_j = 0\ (j \neq o)$, $\boldsymbol{s}^- = \boldsymbol{0}$, $\boldsymbol{s}^+ = \boldsymbol{0}$, and hence we have $\eta^* \geq 1$.

Let us assume that [CCR-O] has a finite optimum η^* $(< \infty)$ and $\boldsymbol{y}_o$ contains negative elements. Then, without losing generality, we can assume that $y_{1o} < 0$. The corresponding constraint in [CCR-O] is:

$$\eta y_{1o} = y_{11}\lambda_1 + \cdots + y_{1n}\lambda_n - s_1^+.$$

This constraint cannot be a binding one to η, since $y_{1o} < 0$ and $s_1^+ \geq 0$, i.e., we can enlarge η as large as possible by enlarging s_1^+ together. Therefore, we can delete this constraint from the above LP [CCR-O]. This leads to the statement that the magnitude of the negative output y_{1o} is irrelevant to the efficiency score η^* of the DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$, if η^* is finite.

Thus, in order to overcome this difficulty, we need to translate the negative output to positive by adding a positive common number to all outputs in the same output item. However, this operation deforms the production possibility set, and usually produces a wrong efficiency score, except for the input-oriented BCC case (see Ali and Seiford (1990)).

3 A scheme using the non-radial SBM approaches

The slacks-based measure (SBM) evaluates the relative efficiency of DMU $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ by solving the following program with variables $\boldsymbol{\lambda}, \boldsymbol{s}^-$ and $\boldsymbol{s}^+$.

$$[\text{SBM}] \quad \rho^* = \min \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}}} \qquad (5)$$

subject to

$$\boldsymbol{x}_o = X\boldsymbol{\lambda} + \boldsymbol{s}^- \qquad (6)$$

$$\boldsymbol{y}_o = Y\boldsymbol{\lambda} - \boldsymbol{s}^+ \qquad (7)$$

$$\boldsymbol{\lambda} \geq \boldsymbol{0},\ \boldsymbol{s}^- \geq \boldsymbol{0},\ \boldsymbol{s}^+ \geq \boldsymbol{0}. \qquad (8)$$

The [SBM] encounters trouble with non-positive y_{ro}, since the term s_r^+/y_{ro} is indefinite if $y_{ro} = 0$ and negative if $y_{ro} < 0$.

3.1 A new scheme for resolving the non-positive problem

In an effort to resolve this problem with the case $y_{ro} \leq 0$, we modify the term s_r^+/y_{ro} in the objective function as follows.

First, we define y_r^M and y_r^m by

$$y_r^M = \max_{j=1,\dots,n} \{y_{rj} | y_{rj} > 0\} \qquad (9)$$

$$y_r^m = \min_{j=1,\dots,n} \{y_{rj} | y_{rj} > 0\}. \qquad (10)$$

If the output r has no positive element, we define

$$y_r^M = y_r^m = 1. \qquad (11)$$

Secondly, we replace the term s_r^+/y_{ro} in the objective function in the following way. (We notice that we never change the value y_{ro} in the constraint.)

[Case 1] If $y_r^M > y_r^m$, we replace the term by

$$s_r^+ \Big/ \frac{y_r^m(y_r^M - y_r^m)}{y_r^M - y_{ro}}. \qquad (12)$$

[Case 2] If $y_r^M = y_r^m$, we replace the term by

$$s_r^+ \Big/ \frac{(y_r^M)^2}{B(y_r^M - y_{ro})}, \qquad (13)$$

where B is a large positive number, e.g., B=100.

In any case, the denominator is positive and strictly less than y_r^m. Furthermore, it is in reverse proportion to the distance $y_r^M - y_{ro}$. Thus, this scheme takes into account the magnitude of the non-positive output. The score thus obtained is units invariant, i.e., it is independent of the units of measurement used.

In order to demonstrate the suitability of this scheme, we examine the case with two DMUs $(\boldsymbol{x}_1, \boldsymbol{y}_1)$ and $(\boldsymbol{x}_2, \boldsymbol{y}_2)$ that have the same inputs and outputs except for the first element of the outputs, i.e., $\boldsymbol{x}_1 = \boldsymbol{x}_2$ and $y_{r1} = y_{r2}$ $(r = 2, \dots, s)$. As for the first element, we assume that $y_{11} > y_{12}$. Let the optimal objective values for $(\boldsymbol{x}_1, \boldsymbol{y}_1)$ and $(\boldsymbol{x}_2, \boldsymbol{y}_2)$ be ρ_1^* and ρ_2^*, respectively. Then we have the theorem:

Theorem 1 *The optimal objective value ρ_1^* for DMU $(\boldsymbol{x}_1, \boldsymbol{y}_1)$ is greater than that ρ_2^* for DMU $(\boldsymbol{x}_2, \boldsymbol{y}_2)$.*

3.2 The efficient status and projection

A DMU is efficient if and only if $\rho^* = 1$, i.e., $\boldsymbol{s}^{-*} = \boldsymbol{0}$ and $\boldsymbol{s}^{+*} = \boldsymbol{0}$.

A projection to efficient frontiers is given by

$$\boldsymbol{x}_o \leftarrow \boldsymbol{x}_o - \boldsymbol{s}^{-*} \qquad (14)$$

$$\boldsymbol{y}_o \leftarrow \boldsymbol{y}_o + \boldsymbol{s}^{+*}. \qquad (15)$$

4 An illustrative example

We exhibits a simple example with a single input and output.

Table 1: A Sample Negative Data Set and Results

DMU	(I)Input	(O)Output	Score
A	1	3	1
B	1	2	0.667
C	1	1	0.333
D	1	0	0.182
E	1	-1	0.111
F	1	-2	0.074
G	1	-3	0.053

Reference

Ali, Seiford (1990), OR Letters 9, 403-405.

Charnes, Cooper, Rhodes (1978), EJOR 2, 429-444.

Tone (2001), EJOR 130, 498-509.