Simple DEA-game-theoretic Solution under Bi-criteria Environment

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1. Introduction

In the literature on cooperative game theory, there have been many applications to cost or benefit-sharing problems. The proposed DEA-game scheme [1] is in sharp contrast to them, in that we can deal with these problems under multi-criteria environments that are common to real conflicts in our society.

The DEA game has two opposite variations, *max* and *min* games. The one is the selfish or bullish game and the other is the modest or bearish game. We can calculate game-theoretic solutions from the *max* and *min* games separately. The nucleolus allocations of the *max* and *min* games are not always the same, but the Shapley value of the *max* game coincides with that of the *min* game.

Using this new scheme, in the preceding session we proposed the Shapley solution to the membership fee sharing problem in the International Federation of Operational Research Societies (IFORS). Also, we showed that the Shapley value of bi-criteria DEA games is the simple average of two allocations in proportion to each single criterion. In this session, we show that the nucleolus allocations of bi-criteria DEA *max* and *min* games are the same, and this nucleolus solution coincides with the Shapley value of the games. That is, under bi-criteria environment, both the Shapley and nucleolus solutions can be obtained as the simple average of two allocations in proportion to each single criterion. Furthermore, we present the case of applying the assurance region (AR) method to bi-criteria DEA game.

2. Basic DEA game models

Let $X=(x_{ij})\in R_+^{m\times n}$ be the score matrix, consisting of the record x_{ij} of player j to the criterion i. For any coalition $S\subset N=\{1,\cdots,n\}$, a characteristic function c is defined in the following linear programming:

$$\begin{aligned} c(S) &= \max_w \sum_{k\in S}\left(\sum_{i=1}^m w_i x_{ik}\right) \\ \text{s.t.} & \quad \sum_{j=1}^n\left(\sum_{i=1}^m w_i x_{ij}\right)=1 \\ & \quad w_i \ge 0 \quad (\forall i). \end{aligned} \tag{1}$$

Thus, we have a game in coalition form as represented by (N, c). The characteristic function c is sub-additive, and it holds that $c(N)=1$.

On the other hand, we can consider the opposite side of the *max* game (N, c). This is defined by replacing *max* in (1) by *min* as follows:

$$\begin{aligned} d(S) &= \min_w \sum_{k\in S}\left(\sum_{i=1}^m w_i x_{ik}\right) \\ \text{s.t.} & \quad \sum_{j=1}^n\left(\sum_{i=1}^m w_i x_{ij}\right)=1 \\ & \quad w_i \ge 0 \quad (\forall i). \end{aligned} \tag{2}$$

This characteristic function d is super-additive, and it holds that $d(N)=1$.

Between these two games (N, c) and (N, d), it holds that

$$d(S)+c(N\setminus S)=1. \quad \forall S \subsetneq N \tag{3}$$

The equation (3) reveals that (N, c) and (N, d) are dual games, and hence they have the same Shapley allocation. That is, for any player k, it holds that

$$\phi_k(c)=\phi_k(d), \tag{4}$$

where $\phi_k(c)$ and $\phi_k(d)$ are respectively Shapley allocations of games (N, c) and (N, d) for player k.

3. Bi-criteria case

In a bi-criteria (m=2) DEA game, we have the following corollary.

Corollary 1: *In a bi-criteria DEA game, for any* $S\subset N$ *and* $T\subset N$ *with the condition* $S\cap T=\phi$, *it* *holds that*

$$c(S \cup T)+d(S \cup T)=c(S)+c(T)+d(S)+d(T). \quad (5)$$

Based on both equation (4) and Corollary 1, we have the following proposition:

Proposition 1: *In a bi-criteria DEA game, for any player k, it holds that*

$$\phi_k(c)=\phi_k(d)=\frac{c(\{k\})+d(\{k\})}{2}. \quad (6)$$

Similar to the case of Shapley allocation, we have the following proposition:

Proposition 2: *In a bi-criteria DEA game, for any player k, it holds that*

$$\mu_k(c)=\mu_k(d)=\frac{c(\{k\})+d(\{k\})}{2}, \quad (7)$$

where $\mu_k(c)$ *and* $\mu_k(d)$ *are respectively the nucleoli of games (N, c) and (N, d) for player k.*

4. Applying the assurance region method

If all players agree on some preference relations concerning various criteria, we can apply the following "assurance region (AR) method," originally developed in DEA literature. Let the base criterion be w_1. We set constraints on the ratio w_i vs. w_1 (i = 2,・・・, m) as follows:

$$L_i \le \frac{w_i}{w_1} \le U_i, \quad (i=2,\cdots,m) \quad (8)$$

where L_i and U_i denote respectively the lower and upper bounds on the ratio w_i/w_1. These bounds must be set in agreement among all the players. The program (1) is now modified as:

$$\begin{aligned} c(S) &= \max_w \sum_{k\in S}\left(\sum_{i=1}^m w_i x_{ik}\right) \\ \text{s.t.} & \sum_{j=1}^n\left(\sum_{i=1}^m w_i x_{ij}\right)=1 \\ & L_i \le \frac{w_i}{w_1} \le U_i \quad (i=2,\cdots,m) \\ & w_i \ge 0 \quad (\forall i). \end{aligned} \quad (9)$$

As regards the case of applying AR method to bi-criteria DEA game, we have the following corollary:

Corollary 2: *In a bi-criteria DEA game, Corollary 1 holds no matter whether AR method is applied or not.*

Since Corollary 1 is crucial for the proofs of both Proposition 1 and 2, we have now, from Corollary 2, the following corollary.

Corollary 3: *In a bi-criteria DEA game, both Proposition 1 and Proposition 2 hold no matter whether AR method is applied or not.*

5. Conclusion

Allowing number of player to increase renders us hard to obtain Shapley/nucleolus allocation because it involves complex calculations where one needs to determine values for a large number of characteristic functions for all possible potential coalitions. For example, in the case of IFORS problem involving 45 players, the number of all possible potential coalitions is $2^{45} > 35{,}000{,}000{,}000{,}000$!

However, from Proposition 1 and 2, it is clear that in a bi-criteria DEA game, one does not need to consider all the potential coalitions. The Shapley and nucleolus allocations can be obtained as the simple average of two allocations in proportion to each single criterion.

In case of more than two criteria, one needs to compute a game-theoretic solution just from its definition. We have developed software for DEA game, which can directly compute the solution from a score matrix. This code can solve the DEA game involving up to 20 players within a reasonable CPU time on a PC. In case of more than 20 players, however, we need to aggregate them into 20 or less groups. The CPU time increases exponentially with number of players, but linearly with number of criteria.

References

[1] Nakabayashi, K. and Tone, K., (2003): "Egoist's Dilemma: A DEA Game," *GRIPS Research Report Series I-2003-0002*. Under second review for possible publication in *Omega: The International Journal of Management Science.*

[2] Cooper, W.W., Seiford, L.M. and Tone, K., (1999): *Data Envelopment Analysis — A Comprehensive Text with Models, Applications, References and DEA-Solver Software*, Kluwer Academic Publisher.