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Nonlinear Monotone Complementarity Problems in Symmetric Matrices

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1 Introduction

Monotone complementarity problems in symmetric matrices provide a unified mathematical model for various problems arising from statistics and control theory.

We will show the existence and the uniqueness of the weighted central trajectory which converge to a solution of the monotone complementarity problem if the interior of the feasible region of the problem is nonempty.

Denote $\mathcal{M}$: the set of all $n \times n$ real matrices,
$\mathcal{M}^{sym}$: the set of all symmetric real matrices in $\mathcal{M}$,
$\mathcal{M}^{sym}_{+}$: the set of all positive definite symmetric matrices in $\mathcal{M}$,
$\mathcal{M}^{sym}_{++}$: the set of all positive semi-definite matrices in $\mathcal{M}$.

Now we consider a nonlinear monotone complementarity problem in symmetric matrices:

$$\textbf{CP} \;:\; \text{Find an } (\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{M}^{sym}_{+} \times \mathcal{M}^{sym}_{+} \text{ such that } (\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F} \text{ and } \mathrm{Tr}\, \boldsymbol{XY} = 0,$$

where $\mathcal{F}$ is a *maximal monotone* subset of $\mathcal{M}^{sym} \times \mathcal{M}^{sym}$, *i.e.*, $\mathrm{Tr}\,(\boldsymbol{X}-\boldsymbol{X}')(\boldsymbol{Y}-\boldsymbol{Y}') \geq 0$ for every $(\boldsymbol{X},\boldsymbol{Y})$, $(\boldsymbol{X}',\boldsymbol{Y}') \in \mathcal{F}$ (monotonicity) and there is no monotone set which properly contains $\mathcal{F}$.

For $\boldsymbol{X} \in \mathcal{M}$, we write $\boldsymbol{X} \succ \boldsymbol{O}$ if $\boldsymbol{X}$ is positive definite, and $\boldsymbol{X} \succeq \boldsymbol{O}$ if $\boldsymbol{X}$ is positive semi-definite. For the sake of simplicity, we use the following symbols:

$$\begin{aligned}
\mathcal{F}(\boldsymbol{R}_Z) &= \{(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{M}^{sym} \times \mathcal{M}^{sym} : (\boldsymbol{X}-\boldsymbol{R}_X, \boldsymbol{Y}-\boldsymbol{R}_Y) \in \mathcal{F}\},\\
\mathcal{F}_+(\boldsymbol{R}_Z) &= \{(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F}(\boldsymbol{R}_Z) : \boldsymbol{X} \succeq \boldsymbol{O},\ \boldsymbol{Y} \succeq \boldsymbol{O}\},\\
\mathcal{F}_{++}(\boldsymbol{R}_Z) &= \{(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F}(\boldsymbol{R}_Z) : \boldsymbol{X} \succ \boldsymbol{O},\ \boldsymbol{Y} \succ \boldsymbol{O}\},\\
\mathcal{F}^*(\boldsymbol{R}_Z) &= \{(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F}_+(\boldsymbol{R}_Z) : \mathrm{Tr}\,\boldsymbol{XY} = 0\},\\
\mathcal{F}_+ &= \mathcal{F}_+(\boldsymbol{O}) \;\;= \{(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F} : \boldsymbol{X} \succeq \boldsymbol{O}, \boldsymbol{Y} \succeq \boldsymbol{O}\},\\
\mathcal{F}_{++} &= \mathcal{F}_{++}(\boldsymbol{O}) = \{(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F} : \boldsymbol{X} \succ \boldsymbol{O},\ \boldsymbol{Y} \succ \boldsymbol{O}\},\\
\mathcal{F}^* &= \mathcal{F}^*(\boldsymbol{O}) \;\;= \{(\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{F}_+ : \mathrm{Tr}\,\boldsymbol{XY} = 0\},\\
\mathcal{B}_+ &= \{\boldsymbol{R}_Z \in \mathcal{M}^{sym} \times \mathcal{M}^{sym} : \mathcal{F}_+(\boldsymbol{R}_Z) \neq \emptyset\},\\
\mathcal{B}_{++} &= \{\boldsymbol{R}_Z \in \mathcal{M}^{sym} \times \mathcal{M}^{sym} : \mathcal{F}_{++}(\boldsymbol{R}_Z) \neq \emptyset\},\\
\mathcal{B}^* &= \{\boldsymbol{R}_Z \in \mathcal{M}^{sym} \times \mathcal{M}^{sym} : \mathcal{F}^*(\boldsymbol{R}_Z) \neq \emptyset\},
\end{aligned}$$

where $\boldsymbol{R}_Z = (\boldsymbol{R}_X, \boldsymbol{R}_Y) \in \mathcal{M}^{sym} \times \mathcal{M}^{sym}$.

2 Existence and Continuity of Weighted Centers

We consider the following mapping:

$$H(\boldsymbol{X},\boldsymbol{Y}) = \boldsymbol{Y}^{\frac{1}{2}}\boldsymbol{X}\boldsymbol{Y}^{\frac{1}{2}} \quad \text{for} \quad (\boldsymbol{X},\boldsymbol{Y}) \in \mathcal{M}^{sym}_{+} \times \mathcal{M}^{sym}_{+}.$$

Theorem 2.1. *For every $\boldsymbol{A} \in \mathcal{M}^{sym}_{++} \cup \{\boldsymbol{O}\}$ and $\boldsymbol{R}_Z \in \mathcal{B}_{++}$, the set $H^{-1}(\boldsymbol{A}) \cap \mathcal{F}_+(\boldsymbol{R}_Z)$ is nonempty. Moreover, if in addition $\boldsymbol{A} \in \mathcal{M}^{sym}_{++}$, the set $H^{-1}(\boldsymbol{A}) \cap \mathcal{F}_+(\boldsymbol{R}_Z)$ consists of an unique point which is continuous in $\mathcal{M}^{sym}_{++} \times \mathcal{B}_{++}$.* ∎

Theorem 2.2. (1) *$\mathcal{B}_{++}$ is a nonempty open convex subset of $\mathcal{M}^{sym} \times \mathcal{M}^{sym}$.*

(2) *$\mathcal{B}_{++} \subset \mathcal{B}^* \subset \mathcal{B}_+ \subset cl\mathcal{B}_{++}$, where $cl\mathcal{B}_{++}$ denotes the closure of $\mathcal{B}_{++}$.* ∎

Theorem 2.3. *The solution set $\mathcal{F}^*$ of the CP is convex. Moreover, if $\mathcal{F}_{++} \neq \emptyset$, then $\mathcal{F}^*$ is a nonempty and compact convex set.* ∎

3 Trajectory

From Theorem 2.1, for every $\boldsymbol{A} \in \mathcal{M}_{++}^{sym}$ and $\boldsymbol{R}_Z \in \mathcal{B}_{++}$, there exists a unique point $(\boldsymbol{X}_{(A,R_Z)}, \boldsymbol{Y}_{(A,R_Z)})$ such that $(\boldsymbol{X}_{(A,R_Z)}, \boldsymbol{Y}_{(A,R_Z)}) \in H^{-1}(\boldsymbol{A}) \cap \mathcal{F}(\boldsymbol{R}_Z)$ and it is continuous w.r.t. $\boldsymbol{A}$ and $\boldsymbol{R}_Z$. In this section, we consider the trajectory T consisting of $(\boldsymbol{X}, \boldsymbol{Y}, t)$'s such that

$$(\boldsymbol{X}(t), \boldsymbol{Y}(t)) = H^{-1}(\boldsymbol{A}(t)) \cap \mathcal{F}(\boldsymbol{R}_Z(t)) \ (t \in [0,1]), \tag{1}$$

and that

$$\left.\begin{array}{l} \boldsymbol{A}(t),\ \boldsymbol{R}_Z(t) \ \text{ are continuous on } \ t \in [0,1], \\ \boldsymbol{A}(t), \boldsymbol{R}_Z(t) \to \boldsymbol{O} \ \text{ as } \ t \to 0, \\ \boldsymbol{A}(t) \in \mathcal{M}_{++}^{sym} \ \text{ for every } \ t \in (0,1], \\ \boldsymbol{R}_Z(1) \in \mathcal{B}_{++}, \\ \boldsymbol{A}(0) = \boldsymbol{O}, \boldsymbol{R}_Z(0) = \boldsymbol{O}. \end{array}\right\} \tag{2}$$

To solve the CP, we numerically trace the trajectory until t gets sufficiently small.

If in addition, $\boldsymbol{A}(t) = t\boldsymbol{A}(1)$, $\boldsymbol{R}_Z(t) = t\boldsymbol{R}_Z(1)$, we call the solution set of (1) the trajectory T^L with a linear continuation.

Let $\underline{t} = \inf\{t \in (0,1] : (\boldsymbol{A}(t), \boldsymbol{R}_Z(t)) \in \mathcal{M}_{++}^{sym} \times \mathcal{B}_{++}\}$.

Theorem 3.1. *Suppose that the trajectory T is bounded. Then*

(a) *the trajectory T has at least one limiting point as $t \to \underline{t}$,*

(b) $\underline{t} = 0$,

(c) *if $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}, 0)$ is a limiting point of the trajectory T, $(\bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}})$ is a solution of the CP.* ■

Theorem 3.2. *Assume that $\boldsymbol{R}_Z(t) \in \mathcal{B}_{++}$ for all $t \in [0,1]$. Then*

(a) $\underline{t} = 0$,

(b) *the trajectory T is bounded.* ■

Now we are ready to state a natural linear continuation of the CP under some assumptions.

Theorem 3.3. *Let $(\boldsymbol{X}^0, \boldsymbol{Y}^0), \bar{\boldsymbol{R}}_Z \in \mathcal{M}_{++}^{sym} \times \mathcal{M}_{++}^{sym}$ such that $(\boldsymbol{X}^0 - \bar{\boldsymbol{R}}_X, \boldsymbol{Y}^0 - \bar{\boldsymbol{R}}_Y) \in \mathcal{F}$. Let $H(\boldsymbol{X}^0, \boldsymbol{Y}^0) = \bar{\boldsymbol{A}}$ and $\boldsymbol{A}(t) = t\bar{\boldsymbol{A}}$, $\boldsymbol{R}_Z(t) = t\bar{\boldsymbol{R}}$ for all $t \in [0,1]$. Then*

(a) *$(\boldsymbol{A}(t), \boldsymbol{R}_Z(t))$ is a continuous mapping satisfying (2),*

(b) *$\underline{t} = 0$ and the trajectory T^L with the linear continuation is bounded if and only if the CP has a solution.* ■