A Numerical Procedure for the General One-Factor Interest Rate Model

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I. Introduction

In this paper, we restrict our attention to the general one-factor term structure model

$$(1) \qquad dr = a(t)(\phi(t) - r)dt + \sigma(r,t)dB$$

where r is the short rate, B is the Wiener process under a risk-neutral probability measure, and ϕ, a and σ are nonnegative functions. The interest rate process is Markov and possesses the mean-reverting property. The model (1) was considered by Hull and White (1993) where a general numerical procedure is presented involving the use of trinomial trees so that the model is consistent with initial market data.

The purpose of this paper is to develop a numerical procedure for constructing a trinomial tree associated with the general one-factor term structure model (1). It covers not only models that have been suggested in the literature but also new models to be developed. As Hull and White (1993) noted, it is important to be able to test the effect of a wide range of different assumptions by the same procedure, because there is no general agreement on which set of assumptions is best.

II. The General One-Factor Interest Rate Model

The trinomial tree developed by Kijima and Nagayama [3] can be extended to the tree associated with the general one-factor interest rate model (1), although [3] do not mention explicitly. We define

$$(2) \qquad dx(t) = -a(t)x(t)dt + \delta(x(t),t)dB.$$

Suppose $r(t) = \theta(t) + x(t)$. The function $\theta(t)$ is called the *shift function.* Then $\theta(0) = r(0)$ and, from (2), we have

$$dr \quad = \quad (\theta'(t) + a(t)\theta(t) - a(t)r)dt + \delta(x,t)dB.$$

Therefore $\theta(t)$ must satisfy

$$(3) \quad \theta'(t) + a(t)\theta(t) = a(t)\phi(t), \quad 0 \le t \le T.$$

and δ must satisfy

$$(4) \qquad \delta(x(t),t) = \sigma(x(t) + \theta(t), t).$$

We discretize the time parameter t in terms of $\Delta t = T/N$. Let $x_n = x(n\Delta t)$, $a_n = a(n\Delta t)$ and $\theta_n = \theta(n\Delta t)$. Let $\Delta x_n = x_{n+1} - x_n$. Approximation of (2) is given by

$$(5) \Delta x_n = -a_n x_n \Delta t + \sigma(x_n + \theta_n, n\Delta t)\Delta B_n,$$

where $B_n = B(n\Delta t)$ and $\Delta B_n = B_{n+1} - B_n$. Rewriting (5) yields

$$x_{n+1} = (1 - a_n\Delta t)x_n + \sigma(x_n + \theta_n, n\Delta t)\Delta B_n.$$

Let β_0 be a positive number and define the sequence $\{\beta_n\}$ by

$$(6) \beta_n = \beta_{n-1}(1 - a_n\Delta t) = \beta_0 \prod_{k=1}^{n} (1 - a_k\Delta t).$$

Suppose that movements of the discretized stochastic process $\{x_n\}$ follow

$$x_n \begin{cases} \nearrow (1 - a_n\Delta t)x_n + \beta_n & \text{with prob. } p(n) \\ \rightarrow (1 - a_n\Delta t)x_n & \text{with prob. } 1 - 2p(n) \\ \searrow (1 - a_n\Delta t)x_n - \beta_n & \text{with prob. } p(n) \end{cases}$$

(7)

where the probability $p(n, x_n)$ is determined so as to satisfy the variance condition. Namely, since $\sigma(x_n + \theta_n, n\Delta t)\Delta B_n$ has variance $\sigma^2(x_n + \theta_n, n\Delta t)\Delta t$, we must have

$$(8) \quad p(n, x_n) = \frac{1}{2}\left(\frac{\sigma(x_n + \theta_n, n\Delta t)}{\beta_n}\right)^2 \Delta t.$$

Of course, $0 \leq p(n, x_n) \leq \frac{1}{2}$ for all $n \leq N$. This restriction is satisfied as far as

$$\beta_n \geq \sigma(x_n + \theta_n, n\Delta t)\sqrt{\Delta t}. \tag{9}$$

Note that the tree representing movements of $\{x_n\}$ is recombining. This is easily seen from (7) using an induction argument. The values of x_n consist of $\{j\beta_{n-1} \;\; ; \;\; -n \leq j \leq n\}$.

The procedure described above has two main defects in the general framework. The first one is the difficulty of determining the sequence $\{\beta_n\}$ in (6) that must satisfy (9). If the volatility function σ is uniformly bounded and the sequences $\{a_n\}$ and $\{\theta_n\}$ are given, this may be resolved by choosing β_0 appropriately. However, in the general framework, it is not always possible to find such $\{\beta_n\}$. The second problem is that the highest node in the tree possibly non-increases in n, i.e.,

$$n\beta_{n-1} \geq (n+1)\beta_n \tag{10}$$

for some $n < N$. If this happens, the geometry of the tree shrinks as time goes by. It is then imagined that the approximation based on (7) cannot be good.

III. The Proposed Procedure

We define (n, j) as the node for which $t = n\Delta t$ and $x_n = j\beta_{n-1}$, and denote by $x_{n,j}$ the value of x at node (n, j). Since the distributions of ΔB_n in (5) are the same, the higher the volatility, the bigger the movement of x. In order to take this fact into consideration, we introduce a positive integer-valued function φ, called the *step size function*, that determines the multiples of β. Suppose the sequence $\{\beta_n\}$ is determined according to (6) and is fixed. We assume that movements of the discretized process $\{x_n\}$ follow

$$x_n \begin{array}{lll} \nearrow & (1 - a_n\Delta t)x_n + \varphi(n,j)\beta_n & \text{with prob. } p(n) \\ \rightarrow & (1 - a_n\Delta t)x_n & \text{with prob. } 1 - 2p(n) \\ \searrow & (1 - a_n\Delta t)x_n - \varphi(n,j)\beta_n & \text{with prob. } p(n) \end{array}$$

(11)

where $\varphi(n, j)$ and $p(n, j)$ are determined so as to satisfy the variance condition. Namely,

$$p(n,j) = \frac{1}{2}\left(\frac{\sigma(x_{n,j} + \theta_n, n\Delta t)}{\varphi(n, x_{n,j})\beta_n}\right)^2 \Delta t. \tag{12}$$

see (8). Because $0 \leq p(n,j) \leq \frac{1}{2}$, we have

$$\varphi(n,j) \geq \frac{\sigma(x_{n,j} + \theta_n, n\Delta t)\sqrt{\Delta t}}{\beta_n}, \tag{13}$$

which reveals the fact that the higher the volatility the bigger the value of φ. Once φ is determined, the branching probability $p(n, j)$ is given by (12). Since $(1 - a_n\Delta t)x_{n,j} = j\beta_n$, the assumption (11) implies that the node (n, j) branches to the nodes $(n + 1, j + \varphi(n, j))$, $(n + 1, j)$, and $(n + 1, j - \varphi(n, j))$.

To overcome the second defect described by (10), we impose another restriction on the step size function. That is, we require

$$(j + \varphi(n,j))\beta_n \geq j\beta_{n-1},$$
$$(j - \varphi(n,j))\beta_n \leq j\beta_{n-1},$$

where $x_n = j\beta_{n-1}$. Combining, we have

$$\varphi(n,j) \geq \frac{|j| a_n \Delta t}{1 - a_n\Delta t}. \tag{14}$$

The step size function φ can be any positive integer satisfying both (13) and (14) by which the defects of the Kijima-Nagayama procedure described in Section 2 are avoided.

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