A Last Word on the Pseudo-Conservation Law for Discrete-Time Cyclic-Service Systems

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1. Introduction

Boxma and Groenendijk [3] find a pseudo-conservation law for a discrete-time cyclic-service system with batch Bernoulli process (BBP) arrivals, expressing a weighted sum of the mean waiting times at the individual stations in the system as a function of its traffic characteristics. The approach taken in [3] is based on their companion paper [2] treating a continuous-time system with Poisson arrivals. However, the result [3] for the discrete-time system with BBP arrivals, and therefore, the result for the continuous-time system with batch Poisson arrivals (obtained by taking the limit as the slot length tends to zero in [3]) are seen to be incorrect [1,4]. To the best of the author's knowledge, however, there exists no literature discussing how such incorrect argument appears in [3].

The main purpose of the paper is to present a (last form of) pseudo-conservation law for the discrete-time system with BBP arrivals, identifying two errors in Boxma and Groenendijk [3] and making corrections.

2. Model description and notation

Time is divided into slots which are equal to time unity (one) in length for a moment. We consider a multi-queue, single-server system with N infinite queueing capacity stations. Each individual station is visited by the server in a cyclic order according to one among exhaustive, gated, 1-limited and 1-decrementing service strategies [7]. Arrivals occur at the end of a slot. This arrival assumption is called as *late arrival*; see Takagi [8]. Customers arrive at a station according to a batch Bernoulli process [8,9].

Let X_i be the number of arriving customers at station i during a slot with first two moments; $\lambda_i := E[X_i]$, $\lambda_i^{(2)} := E[X_i^2]$. Let H_i be the service time of a customer arriving in station i with first two moments; $h_i := E[H_i]$, and $h_i^{(2)} := E[H_i^2]$. The offered loads are then given as $\rho_i := \lambda_i h_i$, and $\rho = \sum_{i=1}^{N} \rho_i$.

Let S_i be the server switch-over time between stations i and i (mod N) +1 with first two moments s_i, $s_i^{(2)}$. The total switch-over time of the server during a cycle is given as $S := \sum_{i=1}^{N} S_i$ with first two moments s, and $s^{(2)}$.

Let C be the cycle time, i.e. the time between two successive arrivals of the server at a station. The flow balance argument in leads to the mean cycle time as

$$c := E[C] = s / (1 - \rho). \qquad (2.1)$$

Let {e, g, 1*l* , 1d} be the partition of the station index set {1, 2,..., N} where e : = { j | station j is exhaustive}, g : = { j | station j is gated}, 1*l* : = { j | station j is 1-limited}, and 1d : = {j | station j is 1-decrementing}. Denote by A_i the event that the service is provided upon its arrival at station i. The probability of A_i, $Pr[A_i]$, is then given by

$$Pr[A_i] = \lambda_i c = \lambda_i s / (1 - \rho) \quad (i \in 1l), \qquad (2.2)$$

$$Pr[A_i] = \lambda_i (1 - \rho_i) s / (1 - \rho) \quad (i \in 1d). \qquad (2.3)$$

3. The pseudo-conservation law

Let w_i be the mean waiting time of a customer in station i ($1 \le i \le N$). Using the work load decomposition result [3] together with the lumped work load result for the non-vacation (ordinary) system [9], we have

$$\sum_{i=1}^{N} \rho_i w_i = \frac{\rho}{2(1-\rho)} \sum_{i=1}^{N} \lambda_i h_i^{(2)} + \sum_{i=1}^{N} \frac{(\lambda_i^{(2)} - \lambda_i^2 - \lambda_i)}{2(1 - \rho)} h_i^2 + \rho\left[\frac{s^{(2)}}{2s} - \frac{1}{2}\right] + \frac{s}{2(1-\rho)}\left[\rho^2 - \sum_{i=1}^{N} \rho_i^2\right] + \sum_{i=1}^{N} m_i, \qquad (3.1)$$

where m_i denotes the mean amount of work that is left at station i after an arbitrary departure from that station.

Remark 3.1 For a zero switch-over time ($s = 0$, $s^{(2)} / s = 1$) system, (3.1) reduces to the conservation law of Bisdikian [1] and Takahashi & Hashida [9]. In Boxma & Groenendijk [3] the second term of (3.1) is

$$\frac{(\lambda^{(2)} - \lambda^2 - \lambda)}{2(1 - \rho)} h^2$$

where $h := \sum_{i=1}^{N} \frac{\lambda_i}{\lambda} h_i$, $\lambda^{(2)}$ and λ denote the first two moments for lumped arrival batch size ($X := \sum_{i=1}^{N} X_i$). Boxma & Groenendijk [3] miscalculate the mean lumped work load for the non-vacation system. Indeed, their result [3] is valid for a single-station (N = 1) system. ∎

It remains for us to derive the quantity m_i for an individual station. For exhaustive and gated stations it is easily verified that from the definitions of service strategies

$$m_i = 0 \quad (i \in e), \tag{3.2}$$

$$m_i = \lambda_i (c\rho_i) h_i = c\rho_i^2 \quad (i \in g). \tag{3.3}$$

For 1-limited and 1-decrementing stations, somewhat more work is required. Conditioning on A_i, and applying the argument in Shimogawa & Takahashi [6], we obtain

$$m_i = c\rho_i[\lambda_i w_i + \rho_i + \frac{\lambda_i^{(2)} - \lambda_i}{2\lambda_i}] \quad (i \in 1l), \tag{3.4}$$

$$m_i = c\lambda_i(1 - \rho_i)[\rho_i w_i - \{\frac{\lambda_i h_i^{(2)}}{2(1 - \rho_i)}\rho_i - \frac{\lambda_i^{(2)} - \lambda_i^2 - \lambda_i}{2\lambda_i(1 - \rho_i)}\rho_i h_i - \frac{\lambda_i^{(2)} - \lambda_i}{2\lambda_i} h_i \}] \quad (i \in 1d). \tag{3.5}$$

Remark 3.2 In [3], the last term in the curly bracket of (3.5) is missing. Boxma & Groenendijk [3] misunderstand that the mean number of customers who arrive while the server is present at station i but who are still waiting to be served, $E[N_i^+]$, is equal to the mean number of customers left behind by a departing customer in a $\text{Geom}^X/GI/1$ system with arrival batch X_i and service time H_i. However, $E[N_i^+]$ is the mean number of customers left behind by a departing customer in a modified $\text{Geom}^X/GI/1$ system with arrival batch X_i and service time H_i. Here, by *modified* we mean that each busy period begins with only one customer (batch size initiating each busy period is one). Thus, Boxma & Groenendijk's result [3] is valid for a single (non-batch) arrival system. ∎

Substituting (3.2) through (3.5) into (3.1) yields the following result.

Theorem 3.1 (Pseudo-conservation law) For a discrete-time $\text{Geom}^X/GI/1$ type multi-queue system under mixed exhaustive, gated, 1-limited and 1-decrementing service strategies, we have

$$\sum_{i \in e \cup g} \rho_i w_i + \sum_{i \in 1l} \rho_i (1 - \lambda_i c) w_i + \sum_{i \in 1d} \rho_i [1 - \lambda_i c(1 - \rho_i)] w_i$$
$$= \frac{\rho}{2(1-\rho)} \sum_{i=1}^{N} \lambda_i h_i^{(2)} + \sum_{i=1}^{N} \frac{(\lambda_i^{(2)} - \lambda_i^2 - \lambda_i)}{2\lambda_i(1 - \rho)} h_i \rho_i + \rho[\frac{s^{(2)}}{2s} - \frac{1}{2}]$$
$$+ \frac{c}{2}[\rho^2 - \sum_{i=1}^{N} \rho_i^2] + c \sum_{i \in g \cup 1l} \rho_i^2 - \frac{c}{2} \sum_{i \in 1d} \lambda_i^2 h_i^{(2)} \rho_i$$
$$+ c[\sum_{i \in 1l} \frac{\lambda_i^{(2)} - \lambda_i}{2\lambda_i} \rho_i + \sum_{i \in 1d} \frac{\lambda_i^{(2)} - \lambda_i}{2\lambda_i} (1 - \rho_i)\rho_i]$$
$$- \frac{c}{2} \sum_{i \in 1d} (\lambda_i^{(2)} - \lambda_i^2 - \lambda_i) h_i \rho_i . \tag{3.6}$$

Remark 3.3 a) We have so far expressed all quantities in slots with unity slot length. We now assume a slot to be of length Δ. Taking the limit as $\Delta \to 0$ in (3.6), we obtain the continuous-time result of Chiarawongse & Srinivasan [4] for a batch-Poisson arrival $M^X/GI/1$ type multi-queue system. b) For an extension of the pseudo-conservation law to priority systems, see Takahashi and Kumar [10]. ∎

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